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Airport games: The core and its center

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HIGHLIGHTS

- We find a decomposition of the core of an airport game in terms of reduced games.
- We show that the core-center satisfies the basic properties for airport problems.
- The core-center measures how the core changes when a player is cloned.
- We establish a relationship among face games and monotonicity properties.
- We provide a recursive algorithm to compute the core-center through no-subsidy cones.

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ABSTRACT

An approach to define a rule for an airport problem is to associate to each problem a cooperative game, an airport game, and using game theory to come out with a solution. In this paper, we study the rule that is the average of all the core allocations: the core-center (González-Díaz and Sánchez-Rodríguez, 2007). The structure of the core is exploited to derive insights on the core-center. First, we provide a decomposition of the core in terms of the cores of the downstream-subtraction reduced games. Then, we analyze the structure of the faces of the core of an airport game that correspond to the no-subsidy constraints to find that the faces of the core can be seen as new airport games, the face games, and that the core can be decomposed through the no-subsidy cones (those whose bases are the cores of the no-subsidy face games). As a consequence, we provide two methods for computing the core-center of an airport problem, both with interesting economic interpretations: one expresses the core-center as a ratio of the volume of the core of an airport game for which a player is cloned over the volume of the original core, the other defines a recursive algorithm to compute the core-center through the no-subsidy cones. Finally, we prove that the core-center is not only an intuitive appealing game-theoretic solution for the airport problem but it has also a good behavior with respect to the basic properties one expects an airport rule to satisfy. We examine some differences between the core-center and, arguably, the two more popular game theoretic solutions for airport problems: the Shapley value and the nucleolus.

1. Introduction

The airport problem, introduced by Littlechild and Owen (1973), is a classic cost allocation problem that has been widely studied. To get a better idea of the attention it has generated, one can refer to the survey by Thomson (2013). One standard approach to study this problem consists of associating a cooperative game with it and takes advantage of all the machinery developed for cooperative games to gain insights in the original problem. The core, introduced by Gillies (1953), stands as one of the most studied solution concepts in the theory of cooperative games. Its properties have been thoroughly analyzed and, when a new class of games is studied, one of the first questions to ask is whether or not the games in that class have a nonempty core. This is because of the desirable stability requirements underlying core allocations.

Importantly, the cooperative game associated with an airport problem with n agents has a special structure that can be exploited to facilitate the analysis of different solutions. In particular, $2^n - 1$ parameters are needed to define a general n-player cooperative

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game, whereas for an airport game one just needs n. This special structure simplifies the geometry of the core of such games, since they turn to be defined by $2n - 1$ inequality constraints instead of the usual $2^n - 2$.

When the core of a game is nonempty, there is a set of alternatives at which agents' payoffs differ that are coalitionally stable. The core-center (González-Díaz and Sánchez-Rodríguez, 2007) selects the expected value of the uniform distribution over the core of the game: the center of gravity of the core. Therefore, the core-center is an intuitive appealing game-theoretic solution for the airport problem since it represents the "average behavior" of all the stable allocations. There are two important issues concerning the core-center of the airport game that we want to address. First, the computation of the core-center of a general balanced game is very complex. Second, existing rules for the airport problem are evaluated and compared with the core-center in terms of the properties they satisfy or violate. In both cases, the corresponding analysis must be carried out by a detailed examination of the core structure.

The core of an n-player airport game is a $(n - 1)$-dimensional convex polytope, so its $(n - 1)$-Lebesgue measure (its volume) can be seen as the "amount" of stable allocations. Naturally, the mathematical expression of the core-center of an airport game is given in terms of integrals over the core of the game. We provide a decomposition of the core in terms of the cores of the downstream-subtraction reduced games that allows us to find explicit integral formulae for the core-center of an airport game. Building upon this expression, we establish our main result. For each player j, consider the airport problem obtained when agent j makes a clone of himself, that is, replicates his cost. Then, what the core-center assigns to agent j (in the original problem) is the ratio of the number of stable allocations in the game with the clone of player j over the original stable allocations. An important implication of this result is the possibility to implement general volume computation algorithms for convex polytopes to develop methods that effectively compute the core-center of an airport problem. Furthermore, we can easily check that the core-center satisfies many desirable properties: homogeneity, equal treatment of equals, order preservation for contributions and benefits, and last-agent cost additivity among others.

To each agent j, we can associate a face of the core polytope that corresponds to the jth no-subsidy constraint. Each no-subsidy face is the Cartesian product of the cores of two reduced airport games. This particular facial structure of the core of an airport game allows us to derive several results. The rate of change of the number of stable allocations with respect to a parameter cost c_i is proportional to the amount of stable allocations of the j-face game. The variation of what the core-center assigns to player j with respect to the cost parameter c_i depends on the relative position of the core-center of the game and the core-center of the j-face games. As a consequence, we derive a necessary and sufficient condition for the monotonicity of the core-center with respect to the cost parameters in terms of its relative position with respect to the centroids of the no-subsidy faces of the core. Applying this characterization, González-Díaz et al. (2015) show that the core-center satisfies some important monotonicity properties: individual cost monotonicity, downstream-cost monotonicity, weak cost monotonicity, and population monotonicity.

The cones rooted at the origin and whose bases are the cores of the no-subsidy face games are called the no-subsidy cones. The core of the airport game can be decomposed as the union of the no-subsidy cones. Using this decomposition, we present the sketch of a recursive algorithm to compute the core-center through the no-subsidy cones. At the end of this recursive process, the core-center is a weighted sum of the core-centers of reduced two-player airport games (geometrically, the midpoints of all the core edges corresponding to the no-subsidy constraints).

In summary, besides the intuition provided by its own definition, the core-center is a well behaved rule and it may be an interesting addition to the list of solutions for the class of airport problems. In that respect, we point out some differences between the core-center and, arguably, the two more popular game theoretic solutions for airport problems: the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969). For instance, we define a natural property, unequal treatment of unequals, and show that, whereas the Shapley value and the core-center satisfy it, the nucleolus does not.

The paper is structured as follows. In Section 2, we present the basic concepts and notations. Then, in Section 3, we obtain the fundamental integral representation of the core-center as the ratio of volumes. The basic properties of the core-center are examined in Section 4. In Section 5, the structure of the faces of the core of an airport game is exploited to obtain a necessary and sufficient condition for the monotonicity of the core-center and an expression that relates the core-center of the game with the centroids of the no-subsidy faces of the core. We conclude in Section 6 with some summarizing remarks and further comments.

2. Preliminaries

We assume that there is an infinite set of potential players, indexed by the natural numbers. Then, in each given problem only a finite number of them are present. Let $\mathcal{N}$ be the set of all finite subsets of $\mathbb{N} = \{1, 2, \ldots\}$.

A *cost game* with transferable utility is a pair (N, c), where $N \in \mathcal{N}$ and $c : 2^N \to \mathbb{R}$ is a function assigning, to each coalition S, its cost $c(S)$. By convention $c(\emptyset) = 0$. Let $\mathcal{V}^N$ be the domain of all cooperative cost games with player set N. Given a coalition of players S, $|S|$ denotes its cardinality. Given $N \in \mathcal{N}$ and $S \subseteq N$, a vector $x \in \mathbb{R}^N$ is referred to as an *allocation* and $x(S) = \sum_{i \in S} x_i$; also, $e_S \in \{0, 1\}^N$ is defined as $e_S^i = 1$ if $i \in S$ and $e_S^i = 0$ otherwise. An allocation is *efficient* if $x(N) = c(N)$. A cost game $c \in \mathcal{V}^N$ is *concave* if, for each $i \in N$ and each S and T such that $S \subseteq T \subseteq N \backslash \{i\}$, $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$.

For most of the discussion and results, we have a fixed n-player set $N = \{1, 2, \ldots, n\}$. A *solution* is a correspondence ψ defined on some subdomain of cost games that associates to each game $c \in \mathcal{V}^N$ in the subdomain a subset $\psi(c)$ of efficient allocations. If a solution is single-valued then it is referred to as an *allocation rule*.

Given a cost game $c \in \mathcal{V}^N$, the *imputation set*, $I(c)$, consists of the *individually rational* and efficient allocations, *i.e.*, $I(c) = \{x \in \mathbb{R}^n : x(N) = c(N)$ and $x_i \leq c(\{i\})$ for all $i \in N\}$. The *core* (Gillies, 1953) is defined as $C(c) = \{x \in I(c) : x(S) \leq c(S)$ for each $S \subset N\}$.

An *airport problem* (Littlechild and Owen, 1973) with set of agents $N \in \mathcal{N}$ is a non-negative vector $c \in \mathbb{R}^N$, with $c_i \geq 0$ for each $i \in N$. Let $\mathcal{C}^N$ denote the domain of all airport problems with agent set N. Throughout the paper, given an airport problem $c \in \mathbb{R}^N$, we make the standard assumption that for each pair of agents i and j, if $i < j$ then $c_i \leq c_j$. An allocation for an airport problem is given by a non-negative vector $x \in \mathbb{R}^N$ such that $x(N) = c_n$. An allocation rule selects an allocation for each airport problem in a given subdomain. A complete survey on airport problems is Thomson (2013).

Given an allocation x, the difference $c_i - x_i$ between agent i's cost parameter and his contribution can be seen as his profit at x. A basic requirement is that at an allocation x no group $N' \subset N$ of agents should contribute more that what it would have to pay on its own, $\max\{c_i : i \in N'\}$. Otherwise, the group would unfairly "subsidize" the other agents. The constraints $\sum_{j \leq i} x_j \leq c_i$ are called the *no-subsidy constraints*.

To each airport problem $c \in \mathcal{C}^N$, one can associate a cost game $c \in \mathcal{V}^N$ defined, for each $S \subseteq N$, by setting $c(S) = \max\{c_i : i \in S\}$; such a game is called an *airport game*. We have denoted by the same letter c both the airport problem and the associated cost game. It