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Increases in risk and demand for a risky asset

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HIGHLIGHTS

- We prove that Central Dominance with $m = 1$ (CD1) is a particular case of Second order Stochastic Dominance.
- We introduce a new class of dominance, named Relative Dominance RD.
- RD is a strict subclass of CD1.
- Strong Risk Dominance, Simple Dominance and Monotone Dominance are particular cases of Relative Dominance.

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ABSTRACT

We first prove the noteworthy fact that Central Dominance with $m = 1$ (CD1), introduced by Gollier (1995), is a particular case of Second order Stochastic Dominance. We then introduce a new tractable class of dominance that we name Relative order and we prove that this class is a strict subclass of CD1. Finally, we show that some known classes of dominance are particular cases of our new class of dominance.

1. Introduction

Many authors have examined the comparative statics effect of a change in risk. It is known that Second order Stochastic Dominance (SSD) is neither necessary nor sufficient to decrease the agent's demand for the risky asset after a shift in the risky asset, in the standard portfolio problem with a risk-free asset and a risky asset. Gollier (2011) proves that an increase in ambiguity aversion does not necessarily imply a reduction in the demand for risky assets.

Gollier (1995) characterized the necessary and sufficient condition on the change in the risky asset to guarantee that all risk averse expected utility agents will increase their demand of the risky asset. This condition is called Central Dominance (thereafter CD).

In this paper we focus on changes in risk which preserve the mean and are consistent with CD.

The paper is organized as follows: first, we present the framework, the decision model, we recall some important and preliminary results, and prove the "unexpected" clear-cut result that CD1 i.e.: Central Dominance with $m = 1$ is a subclass of the famous SSD (Second order Stochastic Dominance) class.

Rothschild and Stiglitz (1970), Machina and Pratt (1997) have shown that a Mean Preserving Increase in Risk (thereafter MPIR) can be obtained by adding a noise to the less risky random variable, or by a sequence of one or more Mean Preserving Spreads (thereafter MPS). We propose a new tractable class of dominance which is consistent with MPIR and CD. We start by a definition of a new dominance that we name *Relative Simple Dominance* (thereafter RSD) and we prove that RSD is a particular case of "Mean Preserving CD1" (thereafter MPCD1). Hence we introduce our new dominance in its full generality, that we name *Relative Dominance* (thereafter RD) and we prove that RD is a strict subclass of MPCD1. Finally, we show that some classes of dominance already existing in the literature are particular cases of our dominance class: on one hand, we show that Strong Increase in Risk introduced by Meyer and Ormiston (1983, 1985) is a particular case of Relative Dominance, on the other hand, the same applied for Simple Dominance introduced by Dionne and Gollier (1992). Moreover, we show that Monotone Mean Preserving Spread about the origin introduced by Quiggin (1992) implies Simple Dominance, hence Relative Dominance.

2. The decision model

We consider a decision maker (DM thereafter) endowed with an initial wealth w. The set $\mathbb{V}$ of assets consists of all bounded real random variables defined on a probability space $(S, \mathcal{A}, P)$ assumed

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to be sufficiently rich to generate any bounded probability law. S is the set of states of nature, $\mathcal{A}$ is a σ-algebra of subsets of S and P is a σ-additive non-atomic probability measure. Any $X \in \mathbb{V}$ is a (real) bounded random variable characterized by a probability distribution, with F_X its cumulative distribution function (i.e.: $F_X(t) = P(X \leq t)$, $\forall t \in \mathbb{R}$).

A sequence $(X_n)_n$ in $\mathbb{V}$ converges in distribution to X, denoted by $X_n \to^d X$, if the sequence of distribution functions F_{X_n} converges to distribution function F_X at every continuity point of the latter.

When X is a finite discrete random variable, it will be denoted as: $\mathcal{L}(X) = (x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$ with $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$, assuming that $x_1 \leq \cdots \leq x_n$.

We consider a strictly risk averse expected utility (EU) decision maker with a von Neumann–Morgenstern utility function $u : \mathbb{R} \longrightarrow \mathbb{R}$, twice continuously differentiable, strictly concave and such that $u'(x) > 0, \forall x \in \mathbb{R}$.

One of the classical and important models in economic theory is: *The Standard Portfolio Problem* (Dionne et al., 1993; Gollier, 2001). The DM has to determine the optimal composition of his portfolio containing a risk-free and a risky asset. The return of the risk-free asset is ρ. The return of the risky asset is a random variable X. Hence the problem of the DM is to determine the optimal composition $(w - \alpha, \alpha)$ of his portfolio, where $w - \alpha$ is invested in the risk-free asset and α is invested in the risky asset. Thus, the payoff function in the last period is:

$$\begin{aligned} W(X, \alpha) &= (w - \alpha)(1 + \rho) + \alpha(1 + X) \\ &= w(1 + \rho) + \alpha(X - \rho). \end{aligned}$$

To simplify the model, we suppose that the risk-free rate $\rho = 0$. Hence, the payoff function is $W(X, \alpha) = w + \alpha X$. The DM chooses α to maximize:

$$U_X(\alpha) = Eu(w + \alpha X) = \int_{\mathbb{R}} u(w + \alpha x)dF_X(x). \quad (1)$$

As Gollier (1995), we restrict attention, to situations where the DM will invest a strictly positive amount α in the risky asset, more precisely when (1) has a unique solution α^* and $\alpha^* > 0$.

For X belonging to $\mathbb{V}$, we denote $[a_X, b_X]$ the support of F_X, and thus confine risky assets to belong to the subset $\mathbb{V}^+$ of $\mathbb{V}$ defined by:

$$\mathbb{V}^+ =: \{X \in \mathbb{V} \mid a_X < 0,\ b_X > 0 \text{ and } E(X) > 0\}.$$

$\mathbb{V}_0^+$ will denote the subset of $\mathbb{V}^+$, containing only finite discrete random variables: $X \in \mathbb{V}_0^+$, if it can be written such that: $\mathcal{L}(X) = (x_1, p_1; \ldots; x_k, p_k; \ldots; x_n, p_n)$ with $x_1 < \cdots \leq 0 < \cdots < x_n$, $p_i > 0$, $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n x_i p_i > 0$.

Assuming the additional Inada condition for u: $\lim_{z\to+\infty} u'(z) = 0$, it is therefore straightforward to check that (1) has a unique solution α_X^* defined by:

$$\begin{aligned} U_X'(\alpha_X^*) &= E(Xu'(w + \alpha_X^* X)) \\ &= \int xu'(w + \alpha_X^* x)dF_X(x) = 0 \end{aligned} \quad (2)$$

and that $\alpha_X^* > 0$, since $U_X'(0) = E(Xu'(w)) = E(X).u'(w) > 0$.

3. Preliminary results: Central Dominance

The objective of many researchers has been to determine the effect of a change in risk on the optimal portfolio.

The problem is to find conditions which guarantee that all risk averse agents will react to the less risky situation by increasing the demand for the asset, i.e: $\alpha_X^* \geq \alpha_Y^*$, after a decrease in risk from Y to X.

Let us present the famous result of Gollier about a new dominance which guarantees that all risk averse expected utility (EU) agents increase their exposure after a shift in distribution.

Gollier (1995) proposes the following definition:

Definition 1. $X, Y \in \mathbb{V}^+ X$ centrally dominates Y if and only if there exists a real scalar $m \in \mathbb{R}^{+*}$ such that $\int_{-\infty}^t xdF_X(x) \geq m \int_{-\infty}^t xdF_Y(x)$, $\forall t \in \mathbb{R}$. It is denoted as $X \succeq_{CD} Y$.

In the particular case of discrete random variables in $\mathbb{V}^+$, Definition 1 translates as follows:

Definition 2. Let $X, Y \in \mathbb{V}_0^+$ such that $\mathcal{L}(X) = (z_1, p_1; \ldots; z_n, p_n)$ and $\mathcal{L}(Y) = (z_1, q_1; \ldots; z_n, q_n)$ with $z_1 < \cdots < z_n$, $p_i \geq 0$, $q_i \geq 0$ and $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$. X centrally dominates Y if and only if there exists a real scalar $m \in \mathbb{R}^{+*}$ such that: $\sum_{i=1}^j z_i p_i \geq m \sum_{i=1}^j z_i q_i$, $\forall j$.

Gollier (1995) proved the following seminal result:

Proposition 1. *$X \succeq_{CD} Y$ is a necessary and sufficient condition to guarantee that all EU risk-averse agents increase their optimal demand for the risky asset when the excess return undergoes a decrease in risk from Y to X.*

4. Second order Stochastic Dominance and Central Dominance

Rothschild and Stiglitz (1970) give the necessary and sufficient condition for X to be preferred to Y by all risk averse *EU* decision makers:

Definition 3. X dominates Y in the sense of Second order Stochastic Dominance ($X \succeq_{SSD} Y$) if:

$$\int_{-\infty}^t F_X(x)dx \leq \int_{-\infty}^t F_Y(x)dx, \quad \forall t \in \mathbb{R}.$$

The following lemma recalls the integration by parts formula, which will prove useful for dealing with general distribution functions.

Lemma 1. *Suppose that G and H are two functions of bounded variations over the interval $[a, b] \subset \mathbb{R}$, then:*

$$\begin{aligned} &\int_{[a,b]} H(t^+)dG(t) + \int_{[a,b]} G(t^-)dH(t) \\ &= H(b^+)G(b^+) - H(a^-)G(a^-). \end{aligned}$$

Particularly, if $G(t) = t$ and H is right continuous, then:

$$\int_{[a,b]} H(t)dt + \int_{[a,b]} tdH(t) = bH(b) - aH(a^-). \quad (3)$$

Proof. For sake of completeness a proof is given in the Appendix. □

We prove now the striking fact that when $m = 1$, Central Dominance is a particular case of Second order Stochastic Dominance.[1] Let us recall that we denote CD1 Central Dominance with $m = 1$.

Theorem 1. *Let $X, Y \in \mathbb{V}$, if $X \succeq_{CD1} Y$, then $X \succeq_{SSD} Y$.*

Proof. Let $X, Y \in \mathbb{V}$ such that $X \succeq_{CD1} Y$.

Let $[a_1, b_1]$ and $[a_2, b_2]$ be the supports of F_X and F_Y respectively.

Take $a = \min(a_1, a_2)$ and $b = \max(b_1, b_2)$ and let $H(x) = F_X(x) - F_Y(x), \forall x \in \mathbb{R}$, $\delta(t) = \int_{-\infty}^t xdH(x)$.

By hypothesis, we have $\delta(t) \geq 0$, $\forall t \in \mathbb{R}$ and we need to prove that setting $G(t) = \int_{-\infty}^t H(x)dx, \forall t \in \mathbb{R}$, we have $G(t) \leq 0$, $\forall t \in \mathbb{R}$.

[1] We are grateful to Christian Gollier who pointed out that our initial proof devoted to the equal mean case can be straightforwardly extended to the general case of eventually different means.