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Existence of an upper hemi-continuous and convex-valued demand sub-correspondence

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HIGHLIGHTS

- Ordering by inclusion and acyclic convexity of a preference relation P are introduced.
- The existence of an ordered by inclusion, open and convex extension of P is proved.
- Ordering by inclusion and convexity of P imply a convex-valued demand.

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ABSTRACT

In this paper we show that a strictly open, non-saturated and acyclically convex preference relation admits an extension which is ordered by inclusion (a weaker property than regularity), strictly open, locally non saturated and convex; in turn, this result permits to prove the existence of an upper hemi-continuous and convex-valued demand sub-correspondence. By directly applying standard fixed-point techniques to these sub-correspondences, it is therefore possible to demonstrate the existence of general economic equilibrium even if consumers' preference relations are not regular.

1. Introduction

In a paper of some years ago Scapparone (1999) showed that a certain property of preferences (which will be named *acyclic convexity* here) is a necessary and sufficient condition for the existence of a regular and convex-valued extension of a strong preference relation. Subsequently, Bossert and Sprumont (2003) and Demuynck (2009) resolved a similar problem with regard to a weak preference relation.

These results are of some importance to general equilibrium theory, since they imply the existence of a convex-valued demand sub-correspondence. As it is well-known, convexity of the set of demanded bundles is indispensable to prove the existence of equilibrium, provided that the fixed point theorem is directly applied to the excess demand correspondence of the economy: see *e.g.* Debreu (1982). On the other hand, if the individual demand correspondences are not convex-valued (because preference relations, although convex, are not regular), it is necessary to employ other and more complex techniques in order to prove the existence of equilibrium: for an extensive review of these results see *e.g.* Sonnenschein (1977). However, if we suppose that each individual preference relation has a regular and convex extension, by a well-known theorem on maximal elements each individual demand correspondence will in turn have a convex-valued sub-correspondence. Therefore, it will be possible to prove again the existence of equilibrium through the traditional method, simply by substituting these sub-correspondences for the original ones in order to build the excess demand correspondence of the economy. Clearly, every equilibrium for this modified economy will be also an equilibrium for the original one.

However, the theorem proved in Scapparone (1999) is not entirely satisfactory from this point of view: in fact, the regular and convex extension whose existence was proved there is not necessarily open, even if the preference relation has this property. Nevertheless, the openness of consumer's preference relations is essential in order to prove that their demand correspondences are upper hemi-continuous, another property which is indispensable for the demonstration of existence of equilibrium: see again Debreu (1982). Therefore, a natural extension of the previous result would be the specification of properties of the preference relation which imply the existence of an upper hemi-continuous and convex-valued demand sub-correspondence. From this point onwards, the demonstration could proceed in the same way we described at the end of the previous paragraph.

In this work we will prove that in order to obtain this result, the regularity of the sought extension is in reality unnecessary: as a

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matter of fact, it is sufficient that the extension is *ordered by inclusion*, a weaker property than regularity, which is equivalent to an assumption about weak preference relations that Bridges (1983) called *pseudo-transitivity*. More precisely, we will prove that if the preference relation is strictly open (a standard reinforcement of the openness), non-saturated and acyclically convex, then it has an ordered by inclusion, strictly open, locally non-saturated and convex extension (Theorem 1). Subsequently, we will demonstrate that these properties imply the existence of an upper hemi-continuous and convex-valued demand sub-correspondence; strict openness permits to extend the proof to a set of price–wealth pairs larger than usual (Theorem 2). This last result makes it possible to apply the more traditional method of proof to a wider class of preference relations, in order to demonstrate the existence of equilibrium.

From a mathematical point of view, the problem studied in this paper can be considered as a generalized case of existence of a continuous selection of a correspondence. In the original version by Michael (1956), the searched selection was in reality a continuous function; afterwards, also the case of multi-valued selections was considered, endowed with various topological and vector space properties in addition to continuity (for an extensive review of the latest results on this topic see *e.g.* Repovš and Semenov, 2014). The main difference is that, in the prevailing literature, the existence of continuous selections is directly inferred from the properties of the original correspondence, which is not necessarily the consequence of a constrained maximization process as in the economic case considered here.

2. Preference relations

In this and in the following section we will study some properties of the preference relation in a very general context; in Section 4 we will specify better the nature of some of the concepts we introduced previously, in order to apply them to demand and general economic equilibrium theories.

We will denote by X the *set of alternatives*, among which a given subject can choose. For the moment we will suppose only that X is a non-empty and convex subset of a real linear topological space S; in particular, we will suppose that S is a Hausdorff space which verifies the first axiom of countability: see *e.g.* Kelley (1955, pp. 50 and 67). We will denote by $\mathrm{int}(X)$ the interior part of X in the topology of S; moreover, we will denote by $\mathcal{A}$ the class of all non-empty and finite subsets of X.

For every subset A of X we will denote by $\mathrm{co}(A)$ the *convex hull* of A, *i.e.* the intersection of all convex subsets of X which include A. It is clear that A is convex if and only if the equality $A = \mathrm{co}(A)$ holds; moreover, for every class $\{A_i : i \in \Im\}$ of subsets of X the following conditions

$$\bigcup_{i\in\Im} \mathrm{co}(A_i) \subseteq \mathrm{co}\left[\bigcup_{i\in\Im} A_i\right] = \mathrm{co}\left[\bigcup_{i\in\Im} \mathrm{co}(A_i)\right] \tag{1}$$

hold. Finally, it is possible to demonstrate the following theorems:

Proposition 1. *For every subset A of X $x \in \mathrm{co}(A)$ holds if and only if there exist a finite set $\{x_1, \dots, x_\ell\}$ of elements of A and a vector $\lambda \in \Re_+^\ell$ such that*

$$\sum_{i=1}^{\ell} \lambda_i \cdot x_i = x \quad \textit{and} \quad \sum_{i=1}^{\ell} \lambda_i = 1. \tag{2}$$

Proof. See *e.g.* Valentine (1964, p. 15). ∎

Proposition 2. *For every convex subset A of X and for every $x \notin A$ the set $\mathrm{co}[A \cup \{x\}] \backslash \{x\}$ is convex.*

Proof. Given any two $y, z \in \mathrm{co}[A \cup \{x\}] \backslash \{x\}$ and any $\mu \in (0, 1)$, basically we must prove that $\mu \cdot y + (1 - \mu) \cdot z \in \mathrm{co}[A \cup \{x\}]$ is different from x. Since the set A is convex, by Proposition 1 there will be two alternatives $y', z' \in A$ and two real numbers $\lambda_y, \lambda_z \in (0, 1]$ such that

$$y = \lambda_y \cdot y' + (1 - \lambda_y) \cdot x \quad \text{and} \quad z = \lambda_z \cdot z' + (1 - \lambda_z) \cdot x. \tag{3}$$

Let us suppose *ab absurdo* that $\mu \cdot y + (1 - \mu) \cdot z = x$ holds: by substituting the relations (3) in the latter equation, we would obtain that

$$x = \frac{\mu \cdot \lambda_y}{\mu \cdot \lambda_y + (1 - \mu) \cdot \lambda_z} \cdot y' + \frac{(1 - \mu) \cdot \lambda_z}{\mu \cdot \lambda_y + (1 - \mu) \cdot \lambda_z} \cdot z'$$

and therefore that $x \in A$, in contrast with our assumptions. Therefore, we conclude that the set $\mathrm{co}[A \cup \{x\}] \backslash \{x\}$ is convex. ∎

We will denote by $P \subseteq X^2$ the *preference relation* of our subject. We will suppose that P is irreflexive: *i.e.* P is a *strong* preference relation, which expresses the superiority of one alternative with respect to another, according to subject's tastes. For every $x \in X$ we will pose $P(x) =: \{y \in X : (y, x) \in P\}$ and $P^{-1}(x) =: \{y \in X : (x, y) \in P\}$.

We will say that the relation P is *acyclic*, if for every set $A \in \mathcal{A}$ the inclusion $A \subseteq \bigcup_{x\in A} P(x)$ does not hold; *transitive*, if for every pair $(y, x) \in P$ the inclusion $P(y) \subseteq P(x)$ holds; *ordered by inclusion*, if for every pair $(y, x) \in X^2$ at least one of the inclusions $P(y) \subseteq P(x)$ and $P(x) \subseteq P(y)$ holds; *regular*, if both the relation P and the corresponding *non-comparability* relation $I \subseteq X^2$, where $(y, x) \in I$ if and only if $(y, x) \notin P$ and $(x, y) \notin P$ hold, are transitive. It can easily be shown that each of these properties implies the previous property, while the converse theorems are not generally true. As we already mentioned in the introduction, the ordering by inclusion is equivalent to the property of *pseudo-transitivity* of the weak preference relation $P \cup I$, according to which if $(z, y) \in P$, $(y, x) \in P \cup I$ and $(x, w) \in P$ hold then we also have $(z, w) \in P$: see *e.g.* Bridges (1983, pp. 27–28). Moreover, the following theorem holds:

Proposition 3. *The relation P is regular if and only if it is acyclic and for every $(y, x) \in I$ the equality $P(y) = P(x)$ holds.*

Proof. See *e.g.* Scapparone (1999, p. 7). ∎

Finally, we will say that the relation P is *non-saturated* if for every $x \in X$ it is $P(x) \neq \emptyset$.

By making use of the topological structure of S, we can attribute further properties to the preference relation; unless they are differently specified, these properties will be defined referring to the relative topology of X. We will say that the relation P is *superiorly open*, if for every $x \in X$ the set $P(x)$ is open; *inferiorly open*, if for every $x \in X$ the set $P^{-1}(x)$ is open; *open*, if P is an open subset in the product topology of X^2; *locally non-saturated*, if for every $x \in X$ it is $x \in \overline{P(x)}$, where $\overline{P(x)}$ is the topological closure of the set $P(x)$; *spacious*, if for every pair $(y, x) \in P$ the inclusion $\overline{P(y)} \subseteq P(x)$ holds. It is easy to see that openness implies both superior and inferior openness, local non-saturation implies non-saturation and spaciousness implies transitivity. Furthermore, the following theorem holds:

Proposition 4. *If the relation P is transitive and inferiorly open then for every pair $(y, x) \in X^2$, such that $y \in \overline{P(x)}$, the inclusion $P(y) \subseteq P(x)$ holds.*

Proof. Taken any $z \in P(y)$, since P is inferiorly open there will be a neighbourhood U of y such that $U \subseteq P^{-1}(z)$. By construction, we can find a $w \in P(x) \cap U$: therefore, by the transitivity of P we conclude that $(z, x) \in P$. ∎