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Optimal risk-sharing under mutually singular beliefs

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HIGHLIGHTS

- An economic setting in which agents have mutually singular beliefs is proposed.
- We demonstrate the existence of equilibrium in this economic setting.
- The characterization of Pareto optimal allocation in our setting is very different from the classical situation, in which all beliefs are mutually equivalent for all agents.

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ABSTRACT

We focus on the situation in which agents might have mutually singular beliefs in a maxmin expected utility framework. We show the existence of an equilibrium under fairly general conditions. We further demonstrate that the characterization of Pareto optimal allocation is significantly different from the classical situation, where all beliefs are mutually equivalent for each agent. Absent aggregate uncertainty, we prove that with common beliefs among agents, any Pareto optimal allocation is a full insurance under the upper capacities for all agents. But the full insurance feature of all Pareto optimal allocations, if true, does not necessarily ensure common beliefs. Moreover, despite agents have sharing beliefs, a full insurance Pareto optimal allocation could be associated with intricate allocation form.

1. Introduction

A principle purpose of research on Knightian uncertainty is to argue that one probability distribution or belief cannot capture the entire uncertainty in all circumstances, and rather, engage a set of probability distributions to represent an agent's uncertainty. Gilboa and Schmeidler (1989)'s maxmin expected utility framework suggests that agents assess uncertainties by evaluating the worst expected utility across all possible probability distributions. Many authors have studied the equilibrium in this maxmin expected utility setting since then. See for instance Dana (2002); Rigotti and Shannon (2012); Rigotti et al. (2008); Dana and Riedel (2013). Yet all of these articles assume that all probability distributions must be equivalent, or equivalently, each agent has mutually equivalent beliefs about the economy. Is this merely an assumption for technical convenience or a crucial assumption with economic significance? We here focus on this assumption and demonstrate its notable implications to the market equilibrium.

It is well recognized since Arrow (1953) that miracles (null events) cannot be distinguished in Savage's classical subjective expected utility theory. For instance, consider a point to be selected at random from a uniform distribution on the unit square and let the event A consist of all the points on the two diagonals and the event B be their intersection. Since both events A and B are treated as miracles, a decision-maker is indifferent between winning a prize if A occurs and winning the same prize if B occurs. However, as argued in Arrow (1953), a decision maker should strictly prefer winning the prize in the event A over winning in the event B. In other words, a decision maker cares about the ranking of null events to some extent. Blume and Brandenburger (1991) define an event E is miracle or null if the decision maker is indifferent among all acts that agree on the complement of E; and propose a non-Archimedean subjective expected utility setting in which the conditioning choices on events of measure zero is possible. As an illustrative example, the events that a die of six faces lands on the edges or the corners are logically possible but might be practically impossible, henceforth, miracles. In an extending state space that includes edges and corners of the die, these events are verifiable. On the other hand, within the realm of the revealed preference framework, these miracles are inadmissible and non-verifiable. Karni (2010) points out the importance to distinguish between impossible events and unverifiable events. As an additional illustra-

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tive example, when a decision maker is asked to choose between betting on the event on the velocity of a particle in a given position takes one of a finite numbers of values and betting on it betting one of these values, the Heisenberg uncertainty principle ensures the nonverifiability of these two conceivable (but null) events.

While null events might be inadmissible, practically impossible, or non-verifiable, it is natural to consider a decision maker who has uncertainty on null events, or in general, mutually singular priors in Gilboa and Schmeidler (1989)'s maxmin expected utility framework. Indeed, when economic agents interpret the latest 2007–2008 financial crisis, clearly, some of them have starkly different views about the occurrence of financial crisis.[1] Different priors on null events are interacted with model uncertainty on the mathematical economic models. In addressing the model uncertainty issues in financial economics, Lo and Mueller (2010) suggest that "model-building in the social sciences should be much less informed by mathematical aesthetics, and much more by pragmatism in the face of partially reducible uncertainty". In addition, Lo and Mueller (2010) propose a taxonomy of uncertainty in which the highest level ∞ uncertainty corresponds to impossible and unverifiable uncertainty. Easley and O'Hart (2009) examine the role of regulation in the presence of ambiguity on the expected means and variances of risky assets.[2] More recently, Epstein and Ji (2013) study a specific economic setting in which the decision maker has mutually singular beliefs about the economic market. Therefore, it is important to examine the implications to market equilibrium when decision makers have mutually singular priors.

This paper offers two chief results. *First*, we characterize conditions under which there exists an equilibrium associated with multiple beliefs among the agents. We demonstrate that mutually equivalent beliefs do not play a crucial role in deriving the existence of an equilibrium, as the maxmin expected utility under a convex and closed set of probability distributions yields inherently a well-performed risk preference, which has been investigated mainly by Mas-Colell and Zame (1991). In other words, it is the nature of the convex and closed set of probability distributions, not the mutually equivalent property among all beliefs, to derive the existence of equilibrium under reasonable assumptions.

Second, we show that the mutually singular beliefs feature *does affect* significantly the characterization of the equilibrium and, in particular, Pareto optimal allocations. Specifically, we demonstrate that absent aggregate uncertainty, and if agents have sharing beliefs, there exist *indeterminate* Pareto optimal allocations. Given the characterization of any Pareto optimal allocation in the classical situation with mutually equivalent beliefs, our result is somewhat surprising. Billot et al. (2000), Dana (2002), Rigotti et al. (2008) prove that in the absence of aggregate uncertainty and all agents have mutually equivalent beliefs, there is common belief if and only if every Pareto optimal allocation is a constant allocation, a so called *full insurance* allocation. Under the same common beliefs assumption and absent aggregate uncertainty, we prove the same characterization theorem subject to a "new" interpretation of the full insurance allocation, that is, the allocation is a constant for each agent under this agent's *upper capacity* resulting from his own set of probability distributions. In the classical situation again, the upper capacity is equivalent to any probability measure in the set of multi-prior beliefs. However, the upper capacity is barely close to any probability measure in the presence of mutually singular beliefs, thus a full insurance Pareto optimal allocation can be extremely complicated. Moreover, despite all Pareto optimal allocations are full insurance in certain circumstances, it is still plausible that all agents have no common beliefs, which is another striking feature different from the classical situation.

Others have analyzed situations with possible mutually singular beliefs. For example, Bewley (2002) and Rigotti and Shannon (2005) examine an incomplete preference in terms of a set of beliefs. By interpreting these beliefs on a larger sample space with additional "model states", one can end up with a set of mutually singular beliefs. By contrast to our presented setting with a complete preference, Bewley (2002) and Rigotti and Shannon (2005) characterize the equilibrium under an incomplete preference in a Knightian decision framework. Furthermore, Kelsey and Yalcin (2007) develop the arbitrage pricing theory under the same incomplete preference.

Our article proceeds as follows. In Section 2 we present the model and in Section 3 we demonstrate some conditions under which the existence of an equilibrium is guaranteed. We specialize two important situations in which there always exist equilibrium and Pareto optimal allocation. In Section 4 we characterize all Pareto optimal allocations associated with non-equivalent beliefs. In addition, we prove the existence of indeterminate Pareto efficient allocations due to agents' mutually singular beliefs. Some comments on our results to compare with other articles are presented in Section 5. In Section 6 we offer our conclusions.

2. The model

Consider a standard one-period pure-exchange economy. The uncertainty in the second period is represented by a state space Ω, and Σ, a σ-algebra on Ω. Let $B(\Omega, \Sigma)$ be the Banach space of real-valued, bounded and measurable functions on Ω, endowed with the sup-norm. Let $ba(\Omega, \Sigma)$ be the space of bounded finitely additive measures on (Ω, Σ) endowed with the weak $*$-topology, $\sigma(ba(\Omega, \Sigma), B(\Omega, \Sigma))$. The norm-dual of $B(\Omega, \Sigma)$ is isometrically isomorphic to $ba(\Omega, \Sigma)$. $B_+(\Omega, \Sigma) \equiv \{x \in B(\Omega, \Sigma) \mid x \geq 0\}$, is the set of nonnegative consumption plans in this economy.

There are n agents in the economy and indexed by $i = 1, \ldots, n$. Agent i's initial endowment $\omega_i \in B_+(\Omega, \Sigma)$, and $\omega \equiv \sum_{i=1}^n \omega_i$ is the aggregate endowment. We assume that each agent's consumption space is $B_+(\Omega, \Sigma)$. For a $x_i \in B_+(\Omega, \Sigma)$, x_i is an interior if $x_i(s) > 0$ for all state s. We write $x_i \in intB_+(\Omega, \Sigma)$ for an interior consumption x_i. The element of the $B_+(\Omega, \Sigma)^n$ is called an allocation. An allocation $x = (x_1, \ldots, x_n)$ is an interior allocation if each x_i is an interior.

Each agent i is risk averse; that he has a monotonic, von Neumann–Morgenstern concave, twice differentiable utility function, $u_i : \mathbb{R}_+ \to \mathbb{R}$, and $u_i(0) = 0$. Agent i is also uncertainty averse; his uncertainty about the beliefs is represented by a set $\mathcal{P}_i$ of probability measures over (Ω, Σ). Throughout this paper we assume that:

Assumption I. The aggregate endowment $\omega \in intB_+(\Omega, \Sigma)$.

Assumption II. Each $\mathcal{P}_i$ is a nonempty, convex, closed subset of $ba(\Omega, \Sigma)$ and all priors in $\mathcal{P}_i$ are σ-additive, for each agent $i = 1, \ldots, n$.

For each agent i with a consumption plan x_i, the utility function V_i is

$$V_i(x_i) := \min_{P \in \mathcal{P}_i} \mathbb{E}_P[u_i(x_i)].$$

Assumption I is a standard one to ensure the existence of equilibrium. Assumption II is also standard in a multi-prior framework. But we do not impose other requirements on the set of beliefs as in literature. For example, Billot et al. (2000) and Rigotti et al. (2008) assume that all priors in $\mathcal{P}_i$ are mutually equivalent; Dana (2002)

[1] According to former Goldman Sachs CFO David Vinor, "we were seeing things that were 25-standard deviation moves, several days in a row", with a probability less than 6.61×10^{-16}; thus this financial crisis is an event which is logically possible but practically impossible. However, Nobel Laureate Joseph Stiglitz argues that those risk models in financial institutions do not focus adequately on the extreme events; or in extreme case, null events. These (financial crisis) events that were supposed to happen once in lifetime of the universe, for some agents, were happening regularly in a short time period like every ten years.

[2] While the ambiguity on the expected mean preserves mutually absolutely continuous priors, it is known that the ambiguity on the volatility leads to mutually singular beliefs. Easley and O'Hart (2009) and Epstein and Ji (2013) investigate the ambiguity on the volatility.