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# Explicit approximate analytic formulas for timer option pricing with stochastic interest rates

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ABSTRACT

The interest rate risk is an important factor in the valuation of timer options. Since the valuation of timer options with interest rate risk is a four-dimensional problem, the dimensionality curse causes tremendous difficulty in finding analytic solutions to the pricing of timer options. In this paper, a fast approximate analytic method is developed to price power style timer options with Vasicek interest rate model. The valuation of timer options with interest rate risk is formulated as a four-dimensional partial differential equation (PDE) using $\Delta$-hedging approach. A dimension-reduction technique is then proposed to reduce the four-dimensional PDE into a two-dimensional nonlinear PDE. A perturbation approach is developed to solve the reduced two-dimensional nonlinear PDEs and then an explicit approximate analytic formula for the timer option is obtained. In particular, explicit approximate analytic formulas for timer options under both Heston and Hull–White models are further derived. Numerical examples of pricing timer options under the above two models are provided. Both the approximate analytic method and the crude Monte Carlo simulation method are used for the examples. The numerical results show that prices of timer options by both methods are close and the approximate analytic method is much faster than the crude Monte Carlo method.


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## 1. Introduction

Volatility is a very important factor in asset price modeling. Financial derivatives on asset volatilities are useful tools for trading, hedging, and risk management. Some information on asset volatility modeling and financial derivatives can be found in Swishchuk (2013), as well as references therein. Timer options are special types of financial derivatives on asset volatilities. Pricing timer options receives significant attention in recent mathematical finance literature. A timer option is similar to a vanilla European option with a random maturity date which is specified as the first time when the accumulated variance of the underlying asset price reaches a given budget level. As discussed in Sawyer (2007), this type of product is designed to give investors more flexibility and to ensure that they do not overpay for an option. In 2007 a timer option was first traded by Société Générale Corporate and Investment Banking (SG CIB). Due to the complexity, timer options were first sold to sophisticated investors such as hedge funds. These options are now becoming more and more popular.

In fact, timer options first appeared in the academia, such as the "mileage option" studied by Neuberger (1990) and the continuous time models discussed by Bick (1995) many years ago, when such securities did not exist in financial market. Carr and Lee (2010) studied a robust replication for timer options where the risk-free rate is zero.

For the Hull–White stochastic volatility model (see Hull & White, 1987), Geman and Yor (1993) established an explicit formula for the distribution related to random maturity date using some remarkable analytical properties of Bessel processes. In Saunders (2011) an asymptotic expansion was developed for fast mean reverting stochastic volatility models of timer options. Monte Carlo methods were used to simulate the prices of timer options for constant interest rates, e.g., Li (2010) and Bernard and Cui (2011). More attractively, Bernard and Cui (2011) proposed a technique of time change to reduce the computational cost of a single timer option from many hours to a few minutes. Liang, Lemmens, and Tempere (2011) used path integral technique developed in quantum field theory to evaluate timer options. Li and Mercurio (2013) developed closed-form approximation to timer option prices under general stochastic volatility models. Li (2013) obtained a Black–Scholes–Merton type formula for pricing timer options using joint distribution of the first-passage time of the realized variance and the corresponding variance characterized by Bessel processes with drift.

To the best of our knowledge, it is assumed that interest rates are constants for all the studies on timer option pricing reported in the literature. However, the effect of the correlations between the variance process and the interest rate process is also an important risk factor for timer options as pointed out in Bernard and Cui (2011). The problem of pricing timer options becomes very difficult for the case with stochastic interest rates due to the complexity of the correlations among interest rate, volatility, underlying asset and random maturity. In this paper, we study timer option pricing with stochastic interest rates under Vasicek model and develop an efficient method, the approximate analytic method, to evaluate timer options with power-type payoffs and small volatility of volatility in the stochastic volatility models. We first derive a four-dimensional partial differential equation (PDE) for the timer option using $\Delta$-hedging approach. Then a dimension-reduction technique is proposed to reduce the four-dimensional PDE into a two-dimensional nonlinear PDE. A perturbation approach, which was used by Li and Mercurio (2013) for obtaining a solution of the two-dimensional PDEs for timer options with constant interest rates, is developed to solve the reduced two-dimensional nonlinear PDEs and obtain an approximate analytic formula for the timer options. The general form of the approximate analytic formula contains some definite integrals and a parameter which is determined by a simple nonlinear equation (see Theorem 3.3). For most popular volatility models like Heston models and Hull–White models, these integrals can be calculated explicitly. Alternatively these integrals can be calculated by numerical quadrature rules. The parameter involved in the formula can be fast calculated by solving the nonlinear equation using Newton's iteration method. In particular, explicit approximate analytic formulas for timer options under both Heston and Hull–White models are further derived. Numerical results in comparing to the crude Monte Carlo simulation show that the approximate analytic formulas are accurate and much faster than the crude Monte Carlo method.

The organization of the rest of this paper is as follows. First, a four-dimensional partial differential equation (PDE) satisfied by the timer option is derived using $\Delta$-hedging approach in Section 2. Next, the resulting four-dimensional PDE is reduced to a two-dimensional nonlinear PDE