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Ranking opportunity sets on the basis of similarities of preferences: A proposal

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HIGHLIGHTS

- Ranking alternative sets in terms of a lexicographic order.
- Intensity of preference over alternatives may affect an individual's choice.
- Axiomatic characterization of an ordering rule.
- Taking into account the similarities of alternatives.

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ABSTRACT

This paper provides a different proposal for ranking sets of alternatives in terms of a lexicographic rule. We discuss how intensity of preference over alternatives may affect an individual's choice out of the available set of alternatives. We provide an axiomatic characterization of an ordering rule for ranking sets of available alternatives, taking into account the similarities of the elements within each set.

1. Introduction

The aim of this paper is to explore the use of information on the preference intensity of individual alternatives, to derive a ranking over sets of available alternatives based on their similarity. The approach of this paper resembles the analysis of Pattanaik and Peleg (1984) and Pattanaik and Xu (2000).

In our context, we suppose that information about the intensity of preference is available and that it allows us to pair alternatives that have a similar level of preference. Therefore, the notions of being 'slightly worse' or 'much better' can be expressed adequately. This can be done, for instance, by using an utility function representing the preference relation and assuming a richer information structure than just ordinality, that is a function that ranks the alternatives whilst at the same time measuring them according to the preference of the agent. Cardinal utility theory states that the satisfaction gained from a particular good or service can be measured, it allows us to define a binary relation between similar alternatives: we consider that two alternatives are similar when they differ by a grade of utility that we feel is inconsiderable.

Actually, we can abstract the idea of being alike in some respect by grouping the objects that are alike together or by making a statement that two given objects are alike or different. These two notions can be abstracted into the idea of partition; the formal status of that assumption is analogous to the similarity-based partition used in Pattanaik and Xu (2000).

The central question is: if preferences are measurable, how can we determine a preference extension over sets of alternatives bearing in mind similarities between individual alternatives? Our work is motivated by several possible applications. The concept of measurable intensity of preference is quite natural, in order to compare sets of alternatives, as a basis for theories concerning money values. As a natural way, we show our degree of preference over alternatives. Additionally, when we must choose between two large groups of alternatives we usually join up the alternatives we feel are similar. In this simple way we can easily choose the best group to compare the original sets.

Consider a consumer ranking opportunity sets. The choice situation may be as common as a situation when an individual selects, for instance, a future university or a restaurant for dinner. It is natural to assume that the decision-maker, when comparing, identifies each university with the set of its most significant professors and pays special attention to the quantity of those that are the most qualified. In the second situation, it seems intuitively plausible to argue that each restaurant may be identified with the set of most significant dishes from their menus and special attention is paid to the number of those that are the most appealing ones. Later on, it is possible to compare the restaurants.

With this intuition, in this paper we propose applying a lexicographic rank-ordered rule to the groups of alternatives that characterize the sets from point of view of similarity. More specifically,

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given two sets of available alternatives, we use a type of lexicographic rule like the one used in Pattanaik and Peleg (1984), over two new sets of alternatives. These new sets of alternatives are constructed taking a partition based on the similarities of these alternatives to choose the best elements in each piece as well as all the analogous ones to the best element of the entire set.

The paper is organized as follows. In the next section, the basic notations and definitions are presented. Section 3 establishes the rule we propose. The main result of the paper, a characterization of the proposed rule, is stated and proved in Section 4 and, finally, Section 5 provides some concluding remarks.

2. Notations and definitions

Consider an agent who may be faced with different sets of alternatives. Let X be the set of basic alternatives available to the agent, assumed to be finite. Let Z be the set of all non-empty subsets of X that are referred to as opportunity sets. The elements of Z are the sets of available alternatives that the agent may be faced with.

Let R be a complete, reflexive and transitive binary relation defined over X. The indifference relation associated with R is denoted by I, while the strict preference relation is denoted by P. For all $A \in Z$ and all $x \in X$, we write xPA to indicate that xPa, $\forall a \in A$, and we write APx to indicate that aPx, $\forall a \in A$. To simplify our exposition, we assume that R is a linear (that is, antisymmetric) ordering. However, this is not an essential assumption, and it is easy to check that, if this assumption is relaxed, the results and the proofs in this paper would only suffer minor modifications.

Let $u: X \to [0, 1]$ be a cardinal utility function for the preference relation R, that is, an utility function that preserves preference orderings ($u(x) \geq u(y)$ iff xRy) uniquely up to positive affine transformations.

2.1. The similarity relation

We assume that preference can be measured by the agent, so those alternatives that differ to a certain degree in utility may be appreciated as similar. This similarity can be represented by a number $r_u > 0$ determined by the capability of the agent to discern, that is, for all $x, y \in S$, xSy is to be interpreted as 'x is similar to y' iff $|u(x) - u(y)| \leq r_u$. Otherwise, we denote $\neg xSy$, that is to be interpreted as 'x is dissimilar to y'. For all $A \in Z$, we say that A is homogeneous iff, for all $a, a' \in A$, aSa'.

Note that we can assume that the similarity relation is insensitive to the particular cardinal utility representation chosen. The number r_u, used to define the similarity relation, is determined by the agent for the utility function used. If we consider another cardinal utility function, obtained from u by a linear positive transformation $v = a + bu$ (with $b > 0$), then $r_v = br_u$ generates the same similarity relation.

Thus, S is a reflexive, symmetric, but not necessarily transitive binary relation defined over X. Thus 'being similar to' is not necessarily an equivalence relation but, it could be defined as a special partition based on this binary relation. For all $A \in Z$, we will write $\phi(A) = \{A_1, \ldots, A_k\}$ where $A_1 = \{a \in A : aSa_1\}$, a_1 being the best element of A according to R; $A_2 = \{a \in A : aSa_2\}$, a_2 being the best element in $A \setminus A_1$, and so on. Since R is a linear ordering, all of these elements a_i are well-defined and unique. Throughout the paper, we assume that the best elements of the partitionings of any set A are numbered in this way and we refer them as the 'significant' elements of A.

It is quite important to observe that, from a formal viewpoint, not only is the existence of a cardinal utility not necessary to construct the similarity relation S, but also less restrictions on the preference relation are required if we consider it as a primitive notion. We could define, with more generality, the similarity relation as the relation that verifies the properties we describe below, but we think that constructing the similarity relation by using a cardinal utility may help us to understand easily and intuitively this concept and its properties.

In this way, for all $A \in Z$, the best smallest similarity-based partition of A, $\phi(A)$, could be defined as the unique class $\{A_1, \ldots, A_k\}$ that verifies the following statements:

- $A_1, \ldots, A_k$ are all non-empty and homogeneous subsets of A
- $A_1 \cup \cdots \cup A_k = A$
- $A_1, \ldots, A_k$ are pairwise disjoint
- if $i < j$, then xPy, for all $x \in A_i$, $y \in A_j$
- for each $y \in A_j$, if $i < j$, then there exists $x \in A_i$ such that $\neg xSy$.

For example, let $A = \{x, y, z\}$ such that $xPyPz$, xSy, ySz and $\neg xSz$, then $\phi(A) = \{\{x, y\}, \{z\}\}$.

3. The similarity-based lexicographic rule

The elements of Z are interpreted as possible opportunity sets that may be available to an individual but, how do individuals rank these opportunity sets based on their preferences for X? Consequently, the problem to be analyzed in this paper is how to establish an ordering (that is a reflexive, transitive and complete binary relation) $\succeq$ over Z. This ordering is interpreted as the agent's preference ranking over opportunity sets. This ordering will be an extension of R to Z, that is, for all $x, y \in X$, xRy iff $\{x\} \succeq \{y\}$. The asymmetric and the symmetric factors of $\succeq$ are denoted by $\succ$ and $\sim$, respectively.

Let $A \in Z$, we have $\phi(A) = \{A_1, \ldots, A_m\}$ and we denote

$$v(A) = (\overbrace{u(a_1), \ldots, u(a_1)}^{k}, u(a_2), \ldots, u(a_m), \overbrace{2, \ \ldots \ , 2}^{n-m-k+1}) \in \mathcal{R}^n$$

with $k = \#A_1$ and $n = \#X$.

We rank the sets by using a lexicographic ordering over the significant elements, leaving out all the similarities except the alternatives that are similar to the best element in each set. In this way, we identify the set A with the vector of $\mathcal{R}^n$, $v(A)$, composed of the utility values of the significant elements of A. The utility value of the best element of A is repeated as many times as the number of elements that are similar to the best. We complete the last coordinates with the number 2 to award in some way the smaller set with respect its supersets. Obviously, the number 2 is arbitrary, any number larger than 1 will do.

Definition 1. $\succeq$ will be called the SL-ordering, for all $A, B \in Z$, $A \succeq B$ iff $v(A) \geq_L v(B)$, that means when $v(A)$ is greater than or equal lexicographically to $v(B)$.

We consider the special case where R is a linear ordering and the proof becomes simpler. A linear preference ordering is a reflexive, transitive, complete and antisymmetric binary relation. Note that the best element is well-defined and unique for each subset of X.

Note that with the SL-ordering we try to reflect the preference over opportunity sets of an agent who joins up the alternatives he or she feels are similar and, later, uses the lexicographical ordering, paying special attention to the number of elements similar to the best one. A peculiar feature of the SL-ordering is the highly asymmetric role played by the similarity class of the locally best alternative vis-à-vis other similarity classes. It is precisely that feature that given e.g. $xRyRzRuRv$ with $x \neq y \neq z \neq u \neq v$ and $S = \{(x, y); (y, x); (y, z); (z, y); (u, v); (v, u); (x, x); (y, y); (z, z); (u, u); (v, v)\}$, dictates $\{x, y\} \succ \{x, y, z\} \succ \{y, z\}$ but $\{y, u, v\} \sim \{y, u\}$. This example requires an explicit discussion. In the case where there are alternatives similar to the best, it is important to know what is really the best and the number of elements that are similar to it but, with respect to the alternatives that are not similar to the best, we only pay attention to the best element of each similarity group and how many similarity groups there are, but not to the cardinality of each similarity group.