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# Downsian competition with four parties

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**Abstract**

Our purpose in this article is to study a unidimensional model of spatial electoral competition with four political parties. We assume that the voters are distributed along [0,1] in such a way that the density $\delta$ of this distribution is continuous on [0,1] and strictly positive on (0,1). The parties engage in a Downsian competition which is modeled as a non-cooperative four-person game $\mathcal{G}(\delta)$ with [0,1] as the common strategy set. If $\xi_i$ stands for the $i$th quartile of the above-mentioned distribution, then we prove that $\mathcal{G}(\delta)$ has a pure Nash equilibrium, if and only if $\int_{\frac{\xi_1+t}{2}}^{\frac{t+\xi_3}{2}} \delta(x)\mathrm{d}x \leq \frac{1}{4}$ for every $t \in (\xi_1,\xi_3)$. Moreover, if this condition is satisfied, then $\mathcal{G}(\delta)$ has exactly six pure Nash equilibria, which are characterized by the fact that two of the parties put forward the policy that corresponds to $\xi_1$ and the other two of them put forward the policy that corresponds to $\xi_3$.


*Keywords:* Economic models of elections; Four political parties

*JEL classification:* C72; D72

## 1. Introduction

Models of spatial electoral competition have been widely used in order to study elections. The starting point was an appropriate interpretation of the model introduced by Hotelling (1929) in the context of firms that choose where to position their stores.

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Borrowing after Hotelling (1929), we can introduce a unidimensional model of spatial electoral competition with as many political parties as we want.

We assume that there exist $n$ parties in the political limelight, which we enumerate from 1 to $n$, where $n \in \mathbf{N} \setminus \{0,1\}$. We assume further that the closed interval [0,1] represents both the spectrum of the voters' own ideal policies and the tank from which the $n$ parties draw the policies that they choose to put forward, while the voters are assumed to be distributed along [0,1] in such a way that the density $\delta$ of this distribution is continuous on [0,1] and strictly positive on (0,1). Given points $\alpha$ and $\beta$ of the closed interval [0,1] such that $\alpha < \beta$, the Lebesgue integral $\int_\alpha^\beta \delta(x)\mathrm{d}x$ expresses the percentage of voters whose own ideal policy lies between $\alpha$ and $\beta$. The $n$ parties engage in a Downsian competition in the following sense (see Downs, 1997). Each party puts forward simultaneously with the other parties a certain policy in [0,1] and its goal is to maximize at the elections its share of the votes, under the assumption that a party attracts the votes of those voters whose own ideal policy lies closest to the policy that this party chooses to put forward and if $k$ of the $n$ parties, where $k \in \{1, 2, \ldots, n\}$, choose to put forward the same policy, then at the elections they share equally the votes which are attracted by their common policy. This Downsian competition is modeled as a non-cooperative $n$-person game $\mathcal{G}(\delta, n)$ in strategic form with the parties as players, where for any $i \in \{1, 2, \ldots, n\}$, the set of pure strategies of player $i$ is the closed interval [0,1]. The following results are well known. See Hotelling (1929) and Eaton and Lipsey (1975).

**Theorem A.** *(Hotelling) $(\xi,\xi)$ is the unique pure Nash equilibrium of $\mathcal{G}(\delta,2)$, where $\xi$ is the unique point of [0,1] for which $\int_0^\xi \delta(x)dx = \frac{1}{2}$.*

**Theorem B.** *(Eaton, Lipsey) $\mathcal{G}(\delta,3)$ has no pure Nash equilibria.*

Given any natural number $n \geq 4$ and any continuous function $\delta$: $[0,1] \rightarrow \mathbf{R}$ such that $\delta(x) > 0$ for every $x \in (0,1)$ and $\int_0^1 \delta(x)\mathrm{d}x = 1$, there exists no exhaustive treatment of the problem of finding the pure Nash equilibria of $\mathcal{G}(\delta, n)$. So my purpose in this article is to offer such a treatment for the case $n = 4$ by proving the following result.

**Theorem C.** *If $\xi_i$ is the unique point of [0,1] for which $\int_0^{\xi_i} \delta(x)dx = \frac{i}{4}$, whenever $i \in \{1, 2, 3\}$, then $\mathcal{G}(\delta,4)$ has a pure Nash equilibrium, if and only if $\int_{\frac{\xi_1+t}{2}}^{\frac{t+\xi_3}{2}} \delta(x)\mathrm{d}x \leq \frac{1}{4}$ for every $t \in (\xi_1, \xi_3)$. Moreover, if this condition is satisfied, then the pure Nash equilibria of $\mathcal{G}(\delta,4)$ are exactly the following: $(\xi_1, \xi_1, \xi_3, \xi_3)$, $(\xi_1, \xi_3, \xi_1, \xi_3)$, $(\xi_1, \xi_3, \xi_3, \xi_1)$, $(\xi_3, \xi_1, \xi_1, \xi_3)$, $(\xi_3, \xi_1, \xi_3, \xi_1)$, $(\xi_3, \xi_3, \xi_1, \xi_1)$.*

It is not difficult to see that if the voters are uniformly distributed along the spectrum of their own ideal policies, i.e., if $\delta(x) = 1$ for every $x \in [0,1]$, then $\mathcal{G}(\delta,4)$ has exactly six pure Nash equilibria, which are characterized by the fact that two of the parties put forward the policy that corresponds to the first quartile of this distribution and the other two of them put forward the policy that corresponds to the third quartile of this distribution. On the other hand, it is not difficult to point out distributions of the voters along the spectrum of their own ideal policies for which $\mathcal{G}(\delta,4)$ has no pure Nash equilibria, as is, for example, the case in which $\delta(x) = 2x$ for every $x \in [0,1]$.