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Symmetric product measures: Binomial processes and invariant manifold intersections in dynamical systems

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Abstract

This article proposes the concept of Symmetric Product Measures (SPM) for addressing the geometric characterization of the distribution of the intersections of the stable and unstable manifolds in (hyperbolic) dynamical systems. Prototypical examples of SPM associated with multiplicative binomial processes are addressed, and the application to manifold characterization in non-linear area-preserving systems (Henon-map) is developed. Moreover, the application of SPM to the characterization of measure-theoretical properties of the periodic orbit distribution in hyperbolic dynamical systems is addressed.

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1. Introduction

As fractal geometry has unveiled the possibility of defining and quantifying structural properties of complex and involuted objects possessing some form of

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dilation symmetry (either strict or in statistical sense) [1,2], multifractal analysis has provided a conceptual framework for defining a broad class of almost everywhere singular measures [3,4]. For measures defined on a n-dimensional Lebesgue measurable support possessing non-vanishing Lebesgue measure, almost everywhere singular measures can be defined (in the Radon–Nykodim meaning [5]) as those measures which are not Lebesgue absolutely continuous.

Multifractal analysis has provided a useful tool for analyzing the properties of singular distributions arising from physical applications: the structure of turbulent eddies [6,7], and the properties of invariant measures associated with the long-term properties of dissipative dynamical systems [8,9].

Focusing attention on deterministic dynamical systems, the connection between invariant geometric structures and statistical (measure-theoretical) properties yielded several approaches aimed at describing and quantifying specific features of singular measures associated with the spatial characterization of the manifold structure [10–13].

Motivated by a geometric problem arising in the theory of hyperbolic dynamical systems [14,15], involving the characterization of the intersection set of the stable and unstable manifolds, (and more generally of leaves of the stable and the unstable foliations) [12,13], this article introduces the concept of symmetric product of two measures. The physical and geometrical interest of this concept is related to the definition of a measure for the intersection between stable and unstable leaves of a hyperbolic dynamical system. Specifically, the introduction of the Symmetric Product Measures (SPM) permits to estimate the intersection measure starting from the w-measures (see [12,13], and Section 2) of the stable and unstable foliations (see Section 5 for details).

The article is organized as follows. Section 2 frames the problem of the intersection measures for hyperbolic two-dimensional diffeomorphisms. Section 3 introduces the concept of SPM of two measures and analyzes its properties. Section 4 addresses some applications to binomial multiplicative measures. Section 5 develops the geometric application of the symmetric product measure to the geometric problem of fiber intersections, and to the measure-theoretical characterization of the intersection set of stable and unstable manifolds for a typical non-linear system, namely the Henon map. The implications of this result in the statistical characterization of dynamical systems is addressed in Section 6 which provides numerical and analytical evidence of the connections between the SPM of the w-measures associated with the stable and the unstable foliations and the Bowen measure associated with the distribution of periodic points.

2. Hyperbolic diffeomorphisms and invariant geometric properties

For Axiom A and hyperbolic diffeomorphisms, the global geometric invariant properties refer to the spatial structure of the stable and unstable manifolds associated with the (unstable) periodic points, and more generally to the geometric and statistical properties of the stable and unstable foliations that can be defined starting from the stable and unstable sub-bundles [14,16].