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Advection and dispersion in time and space

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Abstract

Previous work showed how moving particles that rest along their trajectory lead to time-nonlocal advection–dispersion equations. If the waiting times have infinite mean, the model equation contains a fractional time derivative of order between 0 and 1. In this article, we develop a new advection–dispersion equation with an additional fractional time derivative of order between 1 and 2. Solutions to the equation are obtained by subordination. The form of the time derivative is related to the probability distribution of particle waiting times and the subordinator is given as the first passage time density of the waiting time process which is computed explicitly.

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1. Introduction

Continuous time random walks (CTRW) can be used to derive governing equations for anomalous diffusion [1–4]. The CTRW is a stochastic process model for the movement of an individual particle [5,6]. In the long-time limit, the process converges to a simpler form whose probability densities solve the governing equation, leading to a useful model for anomalous diffusion. For a simple random walk with zero-mean, finite-variance particle jumps, the limit process is a Brownian motion $A(t)$ governed by the classical diffusion equation $\partial p/\partial t = \partial^2 p/\partial x^2$ where $p(x,t)$ is the probability density of the random variable $A(t)$. For symmetric infinite-variance jumps (i.e., those with probability density function tails that fall off like $|x|^{-1-\alpha}$ [7] with some index $0<\alpha<2$), the limit process $A(t)$ is an α-stable Lévy motion, and the governing equation becomes $\partial p/\partial t = \partial^\alpha p/\partial |x|^\alpha$ [8]. When waiting times between the jumps are introduced, the limiting process is altered via subordination [9–11]. In this study, we examine the case where the waiting times are independent of jump size, also called an "uncoupled" CTRW. For infinite mean waiting times (whose probability distribution is assumed to decay algebraically with some index $0<\gamma<1$) the limit process is $A(E(t))$ where $E(t)$ is the inverse or first passage time process for the γ-stable subordinator. By virtue of its construction, the process $E(t)$ counts the number of particle jumps by time $t\geqslant 0$, accounting for the waiting time between particle jumps. In the scaling limit, $E(t)$ keeps track of the possibly nonlinear link between real time and the operational time that a particle actually spends in motion. The governing equation becomes $\partial^\gamma p/\partial t^\gamma = \partial^\alpha p/\partial |x|^\alpha$ [2,3]. Some applications [12] seem to indicate a time derivative of order $1<\gamma\leqslant 2$. In this paper, we develop one such equation by extending the CTRW approach to processes with finite-mean waiting times, and compute the distribution of the relevant first passage time process.

2. The model

In the usual CTRW formalism, the long-time limit for the waiting time process is a γ-stable subordinator $D(t)$ [2]. Then the inverse Lévy process $E(t) = \inf\{x : D(x) > t\}$ counts the number of particle jumps by time $t\geqslant 0$, reflecting the fact that the time T_n of the nth particle jump and the number $N_t = \max\{n : T_n \leqslant t\}$ of jumps by time t are also inverse processes. When the waiting times between particle jumps have heavy tails with $0<\gamma<1$ (i.e., infinite mean), subordination of the particle location process $A(t)$ via the inverse Lévy process $E(t)$ is necessary in the long-time limit to account for the amount of time that a particle is not participating in the motion process. The subordination leads to a time derivative of order γ in the governing equation of motion [3]. When waiting times have heavy tails of order $1<\gamma\leqslant 2$, meaning that the probability of waiting longer than t falls off like $t^{-\gamma}$, a different model is needed [13]. In this case, convergence of the waiting time process requires centering to the mean waiting time w, which is not necessary when $0<\gamma<1$. Accounting for this leads to a waiting time process $W(t) = D(t) + wt$ where $D(t)$ is a completely positively