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# Synchronization and partial synchronization of linear maps

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**Abstract**

We study synchronization of low-dimensional ($d = 2, 3, 4$) chaotic piecewise linear maps that are coupled bidirectionally. For Bernoulli maps we find Lyapunov exponents and locate the synchronization transition, which numerically is found to be discontinuous (despite continuously vanishing Lyapunov exponent(s)). For tent maps, a limit of stability of the synchronized state is used to locate the synchronization transition that numerically is found to be continuous. For nonidentical tent maps at the partial synchronization transition, the probability distribution of the synchronization error is shown to develop highly singular behavior. We suggest that for nonidentical Bernoulli maps (and perhaps some other discontinuous maps) partial synchronization is merely a smooth crossover rather than a well-defined transition. More subtle analysis in the $d = 4$ case locates the point where the synchronized state becomes stable. In some cases, however, a riddled basin attractor appears, and synchronized and chaotic behaviors coexist.


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## 1. Introduction

Recently, synchronization of chaotic dynamical systems has been intensively studied [1]. To some extent this is motivated by its numerous experimental realizations in lasers, electronic circuits or chemical reactions [2]. Of interest, however, are also theoretical aspects of this phenomenon. Relatively well understood is the problem of synchronization of identical systems. In this case it is known that synchronization might appear only when the so-called transversal, or conditional [3] Lyapunov exponents are negative [1]. Much less understood, however, remains the problem of partial synchronization that occurs when systems are nonidentical. In such a case chaotic systems do not fully synchronize which causes fundamental problems even with the very detection of synchronization. In addition to studying Lyapunov spectrum, some other methods to detect this transition were proposed. For example it was noticed that at the partial synchronization of continuous dynamical systems driven by common noise, the probability distribution of phase difference changes [4]. One of the difficulties in studying synchronization is the lack of analytical insight into this problem. Consequently, most of the results in this field are based on numerical calculations. It would be desirable to study models that would be exactly solvable if not fully then at least with respect to some properties.

In the present paper we study low-dimensional ($d = 2, 3, 4$) linear maps. Such systems have the advantage that under some conditions their Lyapunov exponents or at least limits of stability of synchronized state can be obtained analytically. More detailed nature of synchronization transition is studied numerically. We show that in the case of discontinuous maps, synchronization is also a discontinuous transition, despite continuously vanishing (as a function of coupling strength) Lyapunov exponent(s). For continuous maps this transition is found to be continuous. We also study partial synchronization of nonidentical maps and its relation with Lyapunov exponents. We show that for tent maps at partial synchronization there are substantial changes in the probability distribution of the synchronization error. But for Bernoulli maps partial synchronization is merely a smooth crossover between synchronized and nonsynchronized regimes. Finally, we examine $d = 4$ maps which might be considered as describing synchronization of two identical two-dimensional systems. This case requires the most subtle analysis, since even in the synchronized state the system is relatively complicated (synchronized manifold is two-dimensional in this case). Using some arguments, based on the spectral properties of Jacobian of this maps, we locate the point where the synchronized state becomes stable. In some cases, this is most likely a global attractor since numerical calculations show that synchronization transition takes place also at this point. However, in some cases, a basin of the synchronized attractor is riddled with basin of another chaotic attractor [5]. We suggests that such a coexistence of synchronized and nonsynchronized chaotic attractors might be a low-dimensional analog of the so-called stable chaos that was found in some spatially extended systems [6].