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Transfer matrices for the partition function of the Potts model on cyclic and Möbius lattice strips

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Abstract

We present a method for calculating transfer matrices for the q-state Potts model partition functions $Z(G,q,v)$, for arbitrary q and temperature variable v, on cyclic and Möbius strip graphs G of the square (sq), triangular (tri), and honeycomb (hc) lattices of width L_y vertices and of arbitrarily great length L_x vertices. For the cyclic case we express the partition function as $Z(\Lambda, L_y \times L_x, q, v) = \sum_{d=0}^{L_y} c^{(d)}\, Tr[(T_{Z,\Lambda,L_y,d})^m]$, where Λ denotes lattice type, $c^{(d)}$ are specified polynomials of degree d in q, $T_{Z,\Lambda,L_y,d}$ is the transfer matrix in the degree-d subspace, and $m = L_x$ ($L_x/2$) for $\Lambda = sq, tri$ (hc), respectively. An analogous formula is given for Möbius strips. We exhibit a method for calculating $T_{Z,\Lambda,L_y,d}$ for arbitrary L_y. Explicit results for arbitrary L_y are given for $T_{Z,\Lambda,L_y,d}$ with $d = L_y$ and $L_y - 1$. In particular, we find very simple formulas the determinant $det(T_{Z,\Lambda,L_y,d})$, and trace $Tr(T_{Z,\Lambda,L_y})$. Corresponding results are given for the equivalent Tutte polynomials for these lattice strips and illustrative examples are included. We also present formulas for self-dual cyclic strips of the square lattice.

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1. Introduction

The q-state Potts model has served as a valuable model for the study of phase transitions and critical phenomena [1,2]. On a lattice, or, more generally, on a (connected) graph G, at temperature T, this model is defined by the partition function

$$Z(G,q,v)=\sum_{\{\sigma_n\}} \mathrm{e}^{-\beta\mathscr{H}} \tag{1.1}$$

with the (zero-field) Hamiltonian

$$\mathscr{H}=-J\sum_{\langle ij\rangle}\delta_{\sigma_i\sigma_j}\ , \tag{1.2}$$

where $\sigma_i=1,\ldots,q$ are the spin variables on each vertex (site) $i\in G$; $\beta=(k_BT)^{-1}$; and $\langle ij\rangle$ denotes pairs of adjacent vertices. The graph $G=G(V,E)$ is defined by its vertex set V and its edge set E; we denote the number of vertices of G as $n=n(G)=|V|$ and the number of edges of G as $e(G)=|E|$. We use the notation

$$K=\beta J,\quad v=\mathrm{e}^K-1 \tag{1.3}$$

so that the physical ranges are $v\geqslant 0$ for the Potts ferromagnet, and $-1\leqslant v\leqslant 0$ for Potts antiferromagnet, corresponding to $0\leqslant T\leqslant\infty$. One defines the (reduced) free energy per site $f=-\beta F$, where F is the actual free energy, via $f(\{G\},q,v)=\lim_{n\to\infty}\ln[Z(G,q,v)^{1/n}]$, where we use the symbol $\{G\}$ to denote the formal limit $\lim_{n\to\infty}G$ for a given family of graphs. In the present context, this $n\to\infty$ limit corresponds to the limit of infinite length for a strip graph of the square lattice of fixed width and some prescribed boundary conditions.

In this paper we shall present transfer matrices for the q-state Potts model partition functions $Z(G,q,v)$, for arbitrary q and temperature variable v, on cyclic and Möbius strip graphs G of the square, triangular, and honeycomb lattices of width L_y vertices and of arbitrarily great length L_x vertices. We label the lattice type as Λ and abbreviate the three respective types as *sq*, *tri*, and *hc*. Each strip involves a longitudinal repetition of m copies of a particular subgraph. For the square-lattice strips, this is a column of squares. It is convenient to represent the strip of the triangular lattice as obtained from the corresponding strip of the square lattice with additional diagonal edges connecting, say, the upper-left to lower-right vertices in each square. In both these cases, the length is $L_x=m$ vertices. We represent the strip of the honeycomb lattice in the form of bricks oriented horizontally. In this case, since there are two vertices in 1–1 correspondence with each horizontal side of a brick, $L_x=2m$ vertices. Summarizing for all of three lattices, the relation between the number of vertices and the number of repeated copies is

$$L_x=\begin{cases} m & \text{if } \Lambda=sq \text{ or } tri \text{ or } G_D\ ,\\ 2m & \text{if } \Lambda=hc\ .\end{cases} \tag{1.4}$$

Here G_D is the cyclic self-dual strip of the square lattice, to be discussed further below.