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Complex networks on hyperbolic surfaces

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Abstract

We explore a novel method to generate and characterize complex networks by means of their embedding on hyperbolic surfaces. Evolution through local elementary moves allows the exploration of the ensemble of networks which share common embeddings and consequently share similar hierarchical properties. This method provides a new perspective to classify network-complexity both on local and global scale. We demonstrate by means of several examples that there is a strong relation between the network properties and the embedding surface.

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1. Introduction

In recent years, it has become increasingly evident that a convenient way to study complex systems constituted of many interacting elements is by associating to each element a node and to each interaction a link between nodes, giving a *network* (or graph). It has been widely noted that complex interconnected structures appear in a wide variety of systems of high technological and intellectual importance. It has been

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pointed out that many such networks are disordered but not completely random [1–4]. On the contrary, they have intrinsic hierarchies and characteristic organizations which are distinguishable and are preserved during the network evolution. In particular, one of the principal feature of these networks is the fact that they are both *clustered* and *connected*. For instance, an individual in a social network has most links within his own local circle, yet each individual in the world is only at a few steps from any other [5]. An example of a completely clustered network is a triangular lattice on a planar surface: in such a network each one of the n nodes is connected with its local neighbors only and the average distance between two individuals scales as $n^{1/2}$. This is a 'large world'. On the other hand, after Erdös and Rényi [6], we know that random graphs are closely connected systems where the average distance scales as $\ln(n)$: a 'small world'. Intermediate structures can be constructed from the planar lattice by adding links between distant nodes making in this way *short cuts*. But such an insertion of a short-cut on the triangular lattice has an important consequence: the network can no longer be drawn on the plane without edge crossings; it is non-planar. The embedding surface must be modified accordingly by creating a 'worm hole' which connects two distant parts of the surface and through which the new link can 'travel'. Such 'worm holes' create short-cut tunnels in the (2D) universe transforming it into a small world.

In this paper we explore the idea of a network that exists, grows and evolves on a hyperbolic surface. The complexity of the network itself is in this way associated with the complexity of the surface and the evolution of the network is now constrained to a given overall topological organization. More precisely, we explore the relation between the properties of a network and its embedding on a surface. An orientable surface (an intersection-free, two-sided, two-dimensional manifold) can be topologically classified in term of its *genus* which is the largest number of non-intersecting simple closed cuts that can be made on the surface without disconnecting a portion (equal to the number of handles in the surface). The genus (g) is a good measure of complexity for a surface: under such a classification, the sphere ($g = 0$) is the simplest system; the torus is the next-simplest ($g = 1$); etc. To a given network can always be assigned a genus: defined to be equal to the minimum number of handles that must be added to the plane to *embed* the graph without edge-crossings. (Accordingly, a planar graph has genus 0 and it can be 'minimally' embedded on the sphere.) Therefore, our approach works in two ways: it is a convenient tool to generate graphs with given complexity (genus) and/or it is a useful instrument to measure the complexity of real-world graphs.

The aim of the present paper is to emphasize *why* it is convenient to study networks in term of their embeddings on a surface. We find several attractive features of this approach: (1) it provides new measures to characterize complexity; (2) it gives a locally planar representation; (3) it provides a hierarchical ensemble classification; (4) it allows the application of topologically invariant elementary moves [7–9]. In addition, let us stress that *any* network can be embedded on a surface, therefore: *why not?*.