Mathematical Social Sciences 76 (2015) 53–63

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

The complexity of power indexes with graph restricted coalitions

Stefano Benati [a,*], Romeo Rizzi [b], Craig Tovey [c]

[a] Dipartimento di Sociologia e ricerca sociale, Università di Trento, Italy
[b] Dipartimento di Informatica, Università di Verona, Italy
[c] School of Industrial and Systems Engineering, Georgia Institute of Technology, United States

HIGHLIGHTS

- We prove that both problems are #P-complete in the strong sense for graphs.
- We prove that both problems are #P-complete in the weak sense for trees.
- We give pseudo-polynomial algorithms for both problems on trees.

ARTICLE INFO

Article history:
Received 20 December 2012
Received in revised form
11 March 2015
Accepted 3 April 2015
Available online 14 April 2015

ABSTRACT

Coalitions of weighted voting games can be restricted to be connected components of a graph. As a consequence, coalition formation, and therefore a player's power, depends on the topology of the graph. We analyze the problems of computing the Banzhaf and the Shapley–Shubik power indexes for this class of voting games and prove that calculating them is #P-complete in the strong sense for general graphs. For trees, we provide pseudo-polynomial time algorithms and prove #P-completeness in the weak sense for both indexes.

1. Introduction

In a weighted voting game, players belonging to the set $V = \{1, \ldots, N\}$ must make a yes-or-no decision on a given issue. Each player i has weight $w_i \in \mathbb{N}$ and the weight of a coalition $X \subseteq V$ is $w(X) = \sum_{i \in X} w_i$. The decision is approved if the weight of a supporting coalition exceeds a threshold Q, e.g. $w(X) \geq Q$. In this game, players with different weights have different power. More precisely, a voter is said to be critical if a losing coalition becomes winning with its support, and player power derives from the number of coalitions in which the player is critical. Using this definition, the well-known Shapley–Shubik and Banzhaf indexes of power are defined, (Banzhaf III, 1965; Shapley and Shubik, 1954; Chalkiadakis et al., 2011) and applied to political analysis, (Bilbao et al., 2002; Taylor and Pacelli, 2008).

Several contributions in the literature have addressed the effective computability of these indexes. Computing the Shapley–Shubik and the Banzhaf indexes in weighted voting games was shown #P-complete in Matsui and Matsui (2001) and Prasad and Kelly (1990). Therefore, no polynomial time algorithm can compute these indexes if $P \neq NP$, and even if $P = NP$ such an algorithm might not exist. For this reason, some authors (Leech, 2003; Castro et al., 2009; Benati and Vittucci Marzetti, 2013) suggest using simulation and heuristic procedures, and some polynomially solvable special cases of voting games are discussed in Chakravarty et al. (2000). Complete enumeration approaches were also experimented with in Klinz and Woeginger (2005). Nevertheless, this negative result holds only in the weak sense, and, as such, pseudo-polynomial algorithms (i.e. procedures whose running time can be bound by a polynomial in *both* $W = \sum_i w_i$ and N) can be implemented, see Matsui and Matsui (2000), Uno (2012), and in practice these algorithms perform very well. Another popular approach uses generating functions, proposed in Algaba et al. (2003) and Bilbao et al. (2000), but generating functions and dynamic programming have the same computational complexity.

These results relate to voting games where all coalitions are possible, but this is a strong assumption in a real life application. For example, if players are parties represented in a Parliament, coalition formation is restricted by several impediments. It is reported that it is unlikely that an extreme right wing party allies with an extreme left wing party without the mediation of the central player, (Pajala and Widgren, 2004). Therefore, to be a reliable measure, the definition of an index of power should

* Corresponding author.
E-mail addresses: stefano.benati@unitn.it (S. Benati), Romeo.Rizzi@univr.it (R. Rizzi), cat@gatech.edu (C. Tovey).

http://dx.doi.org/10.1016/j.mathsocsci.2015.04.001
0165-4896/

consider restrictions on coalition formation. This argument has been raised before: The literature proposes coalition formation restricted by ordered structures and graphs (see the bibliography Grabish, 2009). In this paper, we focus on coalition formation that is restricted by communication graphs, as is proposed in Borm et al. (1992), Myerson (1977) and Owen (1986).

In a communication graph, player i is a node of the graph $G = (V, E)$. There is a link $(i, j) \in E$, if players i and j can communicate. In Myerson (1977), it is shown how to extend the notion of the Shapley value to graph-restricted cooperative games, extending the Banzhaf index to graph-restricted voting games is proposed in Alonso-Meijide and Fiestras-Janeiro (2006) and Grofman and Owen (1982). The previous literature has focused on the axiomatic principles from which the indexes are derived, but the algorithmic issues have yet to be addressed. It is known that calculating power indexes of voting games without any graph topology restrictions is NP-complete or #P-complete, (Prasad and Kelly, 1990; Deng and Papadimitriou, 1994), but little is known when there are constraints on coalitions.

The purpose of the current paper is to study the complexity of computing power indexes on graphs and to provide pseudo-polynomial algorithms for power indexes on trees. Previous research on the subject is found in Fernandez et al. (2002), where the problem of computing the Shapley–Shubik index on graphs is solved with a pseudo-polynomial algorithm in the special case that the graph is a star. Here we prove that both the Shapley–Shubik and the Banzhaf indexes are weakly #P-complete to compute even on the star topology. Hence strongly polynomial time algorithms cannot exist under the usual assumption $P \neq NP$. This justifies the use of pseudo-polynomial algorithms for these indexes. Accordingly, we provide pseudo-polynomial time algorithms for calculating the Shapley–Shubik and Banzhaf indexes on tree topologies. However, for the general graph topology, we prove that both indexes are #P-complete to compute in the strong sense, (Papadimitriou, 1993; Valiant, 1979), ruling out the likelihood that pseudo-polynomial algorithms can calculate them. As a result, we see a hierarchy of difficulty in computing these indices, depending on the topology of the graph. The case in which no topology is defined is equivalent to the case of power on voting games with communication structure that forms the complete graph. The pseudo-polynomial computability of this case matches the easy computability for the case of trees. In the middle of these two extreme cases there are the arbitrary graphs, for which we prove that the complexity class of the problems is higher.

The paper is structured as follows. In Section 2 the main definitions regarding voting games, power and complexity classes are given. Then we analyze what happens when the connections between players form a tree: Section 3 shows the weak #P-hardness of calculating the Banzhaf and the Shapley–Shubik indices, in Section 4 a pseudo-polynomial algorithm is given for calculating the Banzhaf index of power, and in Section 5 an algorithm is given for the Shapley–Shubik index. Next, we analyze the complexity of calculating the indices of power on general graphs, and show the #P-hardness of both models in Section 6.

2. Definitions

Let $V = \{1, 2, \dots, N\}$ be a set of players. Players must make "yes or no" decisions, but every player $i \in V$ controls a different number of votes $w_i \in \mathbb{N}$. A subset $X \subseteq V$ is a *coalition*. The weight of coalition X is $w(X) = \sum_{i \in X} w_i$, $W = w(V)$. The decision is accepted if there is a supporting coalition X that attains or exceeds the quota Q, e.g. $w(X) \geq Q$. What has been described is a classic model of cooperative game theory that is called *weighted voting game*, denoted as $\mathcal{G} = \{Q; w_1, w_2, \dots, w_N\}$.

We study a variant of weighted voting games where players are embedded in a communication network represented by an undirected graph $G = (V, E)$, if the graph is a tree or a forest, then it will be described by $T = (V, E)$. The nodes V represent players and edges $(i, j) \in E$ correspond to links between players i, j. As in Grofman and Owen (1982), we assume that a coalition can form only if players communicate with each other with a direct link or if some other coalition member acts as intermediary. Coalitions are represented by subgraphs: $G[X]$ denotes the subgraph of G induced by $X \subseteq V$, i.e. $G[X] = (X, E[X])$, where $E[X]$ is the set of those edges of G with both endnodes in X. A coalition $X \subseteq V$ is *connected* if $G[X]$ is connected. If the graph is restricted to be a tree $T = (V, E)$, then X is connected if and only if the forest $T[X]$ is itself a tree. The *connected components* of $G[X]$ are the maximal subcoalitions $Y \subseteq X$, such that $G[Y]$ is connected. A weighted voting game in which players are embedded on a graph will be denoted by: $\mathcal{G}_G = \{Q; w_1, w_2, \dots, w_N\}$.

First we consider the Shapley–Shubik index of power. A coalition X is *winning* if it contains a connected sub-coalition $Y \subseteq X$ with $\sum_{i \in Y} w_i \geq Q$. Therefore, the characteristic function of $\mathcal{G}_G$ (or $\mathcal{G}_T$) is:

$$f(X) = \begin{cases} 1 & \text{if } \exists Y \subseteq X : \ w(Y) \geq Q \text{ and } Y \text{ is connected;} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

If $f(S) - f(S - \{i\}) = 1$ then player i is said to be *critical* for coalition S and coalition S is said to be i-critical. Given the fact that a coalition must be winning if it overcomes a threshold and it is connected, player i can be critical because it has an important weight w_i, but also if it connects two losing coalitions: in this case even a player with $w_i = 0$ can be critical. In Myerson (1977), it is shown that the Shapley–Shubik index is the unique function that satisfies a set of axioms, that characterize the value of network restricted cooperative games. For player i, the value is:

$$SS(i) = \sum_{\substack{X \subseteq V \\ X \ni i}} \frac{(|X| - 1)!(N - |X|)!}{N!} \left[f(X) - f(X - \{i\})\right]. \quad (2)$$

When applied to voting games, then the Shapley–Shubik index has a clear political interpretation. It assumes that the yes-or-no decision is always supported by the grand coalition, that is, decisions that are made unanimously by all players V, and then power depends on the fact that players join the coalition one at the time, forming the sequence $S = \pi(1), \dots, \pi(i), \dots, \pi(N)$. For the sequence S, the role of the critical voter is played by $\pi(i)$, such that $f(\{\pi(1), \dots, \pi(i-1)\}) = 0$, but $f(\{\pi(1), \dots, \pi(i)\}) = 1$. The Shapley–Shubik index of power is calculated averaging on all sequences, i.e., assuming that all sequences have the same probability.

The second index of power that can be used on graph voting games is the Banzhaf value. Following (Alonso-Meijide and Fiestras-Janeiro, 2006), we could define the Banzhaf index of player i on graph restricted games as the value:

$$B_1(i) = \frac{1}{2^{N-1}} \sum_{\substack{X \subseteq V \\ X \ni i}} \left[f(X) - f(X - \{i\})\right]. \quad (3)$$

There is an axiomatic justification to function $B_1(i)$, see Alonso-Meijide and Fiestras-Janeiro (2006), but definition (3) assumes that any coalition S, winning or losing, connected or unconnected, may form with equal probability. As previously discussed, unconnected coalitions are not realistic from a political point of view (see Passarelli and Barr, 2007). Indeed, the requirement that coalitions be connected is the defining motivation for that line of research. Therefore we modify (3) by discarding the unconnected sets X, and