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# On neutrality with multiple private and public goods

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HIGHLIGHTS

- We consider an economy with multiple private goods and multiple public goods.
- We obtain an analogue of the neutrality result of Warr (1983) and Bergstrom et al. (1986).
- We provide an algorithm showing how endowment redistributions can be "neutralized" by changes in the amounts contributed to each public good.

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ABSTRACT

We obtain an analogue of the neutrality result of Warr (1983) and Bergstrom et al. (1986) for economies with both multiple private and public goods.



## 1. Introduction

In a classic paper, Warr (1983) demonstrated that, in an economy with one private good and one public good, small redistributions of endowments of the private good among contributors to public good provision will leave the equilibrium total provision of public good provided unchanged.[1] This has become known as "Warr's neutrality result". Bergstrom et al. (1986) provide an elegant formulation of the model and obtain a neutrality result without recourse to first order conditions; they rely on properties resulting from optimization by individual agents. Consider an equilibrium for an economy and a perturbation of endowments with the property that, after the perturbation, every consumer can afford his equilibrium private good allocation. Provided that the perturbation does not change the total amount of endowment of the economy an equilibrium for the economy generates an equilibrium for the perturbed economy in which all consumers have the same private goods allocation and the total public good contribution is unchanged.[2] The assumptions of one-private good, one public good underpins their model and results. Indeed, the authors write "*whether there are less restrictive assumptions that give rise to the same neutrality result is an open question*". We obtain an analogue of the neutrality result of Warr (1983) and BBV for economies with both multiple private and public goods.

Since BBV's celebrated paper there has been a number of insightful papers addressing neutrality issues. To treat multiple private goods, Villanacci and Zenginobuz (2006a) introduce the concept of a private provision equilibrium. This is an analogue of Walrasian equilibrium with private provision of a public good; prices for private goods are Walrasian and individual contributions to public good provision have the property that, in equilibrium, no consumer can benefit by changing his provision. The importance of the one-private-good assumption of Warr and BBV is highlighted by the work of Villanacci and Zenginobuz (2006b, 2007, 2012), which addresses related issues considering multiple private commodities and one public good. They obtain, under a strictly concave production technology assumption, non-neutrality results within several scenarios. To be more precise,

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1 See also Kemp (1984).

2 There are numerous precursors to the BBV model and results; see their paper for references. Many other authors have studied existence of equilibrium and Warr's neutrality result in a variety of contexts; see, for example, Kemp (1984), Itaya et al. (2002), Cornes and Itaya (2010), Silvestre (2012), Allouch (2015) and others.

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Villanacci and Zenginobuz (2006b) show a non-neutrality result (in terms of utilities) even when all households are strict contributors to the public good. To obtain further non-neutrality results, Villanacci and Zenginobuz (2007) consider redistributions of one of the multiple private goods, which is treated as numeraire, among contributors and non-contributors, to the production of the single public good.

The work of Villanacci and Zenginobuz again raises the BBV question: If there are multiple private goods and even multiple public goods under what conditions, if any, will neutrality hold? Is it possible to obtain an analogue of the neutrality result of Warr and BBV without the assumptions of one-private-good, one-public good? The research of Villanacci and Zenginobuz may suggest the answer to the BBV question is completely negative. In this paper we show that, with both multiple private and public goods, if endowments of private good are redistributed in such a way that each consumer can afford his initial equilibrium allocation of private goods at the initial equilibrium prices, then there exists an equilibrium for the post-redistribution economy that satisfies neutrality. We obtain our results in a model allowing multiple private and public goods and with a modification of the equilibrium concept of Villanacci and Zenginobuz (2006a) to allow multiple public goods. The sufficient conditions for our neutrality result rely on both the consumers that contribute to all the public goods and the wealth (value of endowments and profits) that each of them has in the initial equilibrium. Thus, the set of redistributions that allows us to obtain the same equilibrium depends on the initial equilibrium prices and, therefore, differ for each initial situation when there is multiplicity of equilibria. From our Theorem it follows that non-neutrality can only occur when (i) redistributions involve at least one non-contributor for some public good; or (ii) the value of the new endowments at the initial equilibrium prices does not allow consumption of the initial bundle for at least one consumer.

In addition, we introduce a lemma with a constructive proof that provides an algorithm showing how endowment redistributions can be "neutralized" by changes in the amounts contributed to each public good.

## 2. The model

We consider an economy $\mathcal{E}$ with a finite number $L$ of private goods and a finite number $K$ of public goods. There is a set $\mathcal{N}$ of $N$ consumers who individually consume private goods and collectively consume public goods. Each consumer $i \in \mathcal{N} = \{1, \ldots, N\}$ is characterized by her endowment of private goods $e_i \in \mathbb{R}^L_{++}$ and by her preference relation over commodity space $\mathbb{R}^{L+K}_+$. Her preferences are represented by a continuous, concave and monotone-increasing utility function $U_i : \mathbb{R}^{L+K}_{++} \to \mathbb{R}_+$. Define $e = \sum_{i=1}^N e_i$.

There are $K$ firms that produce public goods. A firm $k \in \{1, \ldots, K\}$ is characterized by a production function $F_k : \mathbb{R}^L_+ \to \mathbb{R}_+$ that converts private goods into public good $k$. We assume that each $F_k$ is continuous and concave. Each consumer $i \in \mathcal{N}$ owns a share $\delta_i^k \geq 0$ of the firm $k$'s profit and $\sum_{i=1}^N \delta_i^k = 1$ for each $k$.

## 3. Private provision equilibrium

A *price system* is a vector $(p, q) \in \mathbb{R}^{L+K}_+$, where $p = (p^\ell, \ell = 1, \ldots, L)$ denotes the vector of prices for the $L$ private commodities and $q = (q^k, k = 1, \ldots, K)$ denotes the vector of prices for the $K$ public goods.

Given a price system $(p, q) \in \mathbb{R}^{L+K}_+$, each firm $k$, $k = 1, \ldots, K$, chooses the vector of inputs in $\mathbb{R}^L_+$ that maximizes its profits $\Pi_k(y) = q_k \cdot F_k(y) - p \cdot y$.

Given a price system $(p, q) \in \mathbb{R}^{L+K}_+$ and profits $\Pi_k$ for each firm $k$, consumers choose private goods consumption and voluntary contributions to public good provision. Each consumer takes as given the contributions of the other consumers to public goods. That is, given a vector $(g_j, j \in \mathcal{N}, j \neq i)$ of voluntary contributions, each consumer $i$ solves the problem:

$$\max_{(x,\varrho) \in \mathbb{R}^L_+ \times \mathbb{R}^K_+} \quad U_i(x, g_{-i} + \varrho)$$

$$\text{such that} \quad p \cdot x + q \cdot \varrho \leq p \cdot e_i + \sum_{k=1}^K \delta_i^k \Pi_k,$$

where $g_{-i} = \sum_{j \neq i} g_j$.

**Definition.** A private provision equilibrium for the economy $\mathcal{E}$ is a price system $(p, q)$, a vector of inputs $y = (y_k \in \mathbb{R}^L_+;\ k = 1, \ldots, K)$ for firms, a private commodities allocation $x = (x_i \in \mathbb{R}^L_+;\ i = 1, \ldots, N)$ and an assignment of voluntary contributions $\sum_{i=1}^N g_i = (g^k \in \mathbb{R}_+; k = 1, \ldots, K) = g$ such that,

(i) $(x_i, g_i)$ solves the optimization problem of consumer $i$ for every $i \in \mathcal{N}$.
(ii) $y_k$ maximizes firm $k'$ profit, for every $k$.
(iii) $\sum_{i=1}^N x_i + \sum_{k=1}^K y_k \leq \sum_{i=1}^N e_i$.
(iv) $g^k \leq F_k(y_k)$ for every public good $k$.

We will typically denote a private provision equilibrium by a list $(p^*, q^*, x^*, g^*, y^*)$.

## 4. Neutrality

Let us consider the economy $\mathcal{E}$ described in Section 2. A redistribution of endowments is any allocation $\hat{e}$ such that $\sum_{i=1}^N e_i = \sum_{i=1}^N \hat{e}_i$. Let $\mathcal{E}(\hat{e})$ denote the economy that coincides with $\mathcal{E}$ except for the endowment which is given by $\hat{e}$, a redistribution of $e$.

**Lemma 4.1.** *Let $(p, q, x, g)$ be a vector of prices, allocations, and contributions and let $\Pi_k$ be profits for each firm k, such that $p \cdot x_i + q \cdot g_i = p \cdot e_i + \delta_i^k \sum_{k=1}^K \Pi_k$ for every consumer i. Consider a redistribution $\hat{e}$ of endowments such that $p \cdot x_i \leq p \cdot \hat{e}_i + \delta_i^k \sum_{k=1}^K \Pi_k$, for every i. Define $\Delta e_i$ by $\hat{e}_i = e_i + \Delta e_i$. Then there exists a vector of voluntary contributions $\hat{g}$ such that $q \cdot (\hat{g}_i - g_i) = p \cdot \Delta e_i$ for every consumer i and $\sum_{i=1}^N (\hat{g}_i - g_i) = 0$, that is, $\sum_{i=1}^N g_i = \sum_{i=1}^N \hat{g}_i$.*

Note that after the redistribution, each consumer can afford her initial equilibrium bundle of private goods. Our neutrality result requires more than this. Define a contributing consumer as a consumer whose contribution to every public good is positive in the initial equilibrium. We now restrict redistributions of endowments to redistributions among contributing consumers (that is, $\hat{e}_i = e_i$ for all non-contributing consumers). This is an important assumption for our result below. (see Footnote 4).

**Theorem 4.1** (*Neutrality*)**.** *Let $(p^*, q^*, x^*, g^*, y^*)$ be a private provision equilibrium for the economy $\mathcal{E}$ and let $\Pi_k^*$ denote the equilibrium profits of firm k. Let $\hat{e}$ be a redistribution of endowments such that $p^* \cdot x_i^* \leq p^* \cdot \hat{e}_i + \sum_{k=1}^K \delta_i^k \Pi_k^*$ for every consumer i and $\hat{e}_i = e_i$ for all non-contributing consumers. Then there exists a vector of voluntary contributions to public goods $(\hat{g}_i,\ i = 1 \ldots, N)$ such that $(p^*, q^*, x^*, \hat{g}, y^*)$ is a private provision equilibrium for the economy $\mathcal{E}(\hat{e})$ and $\sum_{i=1}^N \hat{g}_i = \sum_{i=1}^N g_i^*$.*