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# On the Arrow–Hahn utility representation method

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ABSTRACT

In this paper we characterize metric spaces used in Beardon's generalization of Arrow–Hahn utility representation method as generalized Peano continua. For continuous preference relations defined on such metric spaces, we further construct an upper semi-continuous utility function which explicitly depends on the distance.



## 1. Introduction

A binary relation $\succeq$ defined on a set $X$ is called a *preference relation on* $X$ if it is transitive and complete. Given a preference relation $\succeq$ on $X$ and $x_1, x_2 \in X$, we write $x_2 \succ x_1$, if $x_2 \succeq x_1$ and $x_1 \not\succeq x_2$, while we write $x_2 \sim x_1$, if $x_2 \succeq x_1$ and $x_1 \succeq x_2$. Moreover, for every $x \in X$, we set

$$C^+(x) = \{y \in X : y \succeq x\}, \qquad C^-(x) = \{y \in X : x \succeq y\},$$
$$O^+(x) = \{y \in X : y \succ x\}, \qquad O^-(x) = \{y \in X : x \succ y\}.$$

A function $u : X \to \mathrm{R}$ is called *utility function* representing $\succeq$ if, for every $x_1, x_2 \in X$, $x_2 \succeq x_1$ if and only if $u(x_2) \geq u(x_1)$.

When $X$ is endowed with a topology $\tau$, an interesting and widely studied problem is to find conditions on $\tau$ and $\succeq$ which imply the existence of continuous utility functions. Of course a necessary

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condition is that $\succeq$ is *continuous*, that is, for every $x \in X$, $C^+(x)$ and $C^-(x)$ are closed and $O^+(x)$ and $O^-(x)$ are open.

Given a continuous preference relation $\succeq$ on a topological space $(X, \tau)$, a continuous representation of it can be found if either $(X, \tau)$ is second countable (see Debreu, 1964, Proposition 3) or $(X, \tau)$ is separable and connected (see Eilenberg, 1941, Statement 6, p. 43) or $(X, \tau)$ is separable and locally connected (see Candeal et al., 2004, Theorem 1) or $(X, \tau)$ is path-connected and $\succeq$ is countably bounded (see Monteiro, 1987, Theorem 3). We point out that all the above quoted results concern the existence of continuous utility functions but do not provide any effective way of constructing them.

On the contrary, the so-called Euclidean distance approach allows the construction of a utility function from the preference relation in a simpler and more explicit way (see Alcantud and Mehta, 2009, for detailed information and further references on the topic). This method was first introduced by Arrow and Hahn (see Arrow and Hahn, 1971, pp. 82–87) to find a lower semi-continuous (but not necessarily continuous) utility function for continuous preference relations defined on convex and closed subsets of a Euclidean space. The upper semi-continuity can be obtained by assuming that the preference relation is locally non-satiated.

Beardon (1997) shows that such a method continues to work in a more general setting in which, as the author emphasizes, "no assumption about convexity, linear spaces, connectedness is made" (see Beardon, 1997, p. 370). In fact, he proves that Arrow and Hahn's approach provides a lower semi-continuous (but not necessarily continuous) utility function whenever the preference relation is continuous and defined on a metric space whose metric satisfies the following two properties

$$\text{every closed ball is compact,}^{1} \tag{1}$$

$$\begin{aligned} &\text{if } x_1, x_2 \in X \text{ with } x_1 \neq x_2, \text{ then each neighbourhood of } x_2 \\ &\text{contains a point closer to } x_1 \text{ than } x_2. \end{aligned} \tag{2}$$

Moreover, in this setting too, local non-satiation implies upper semi-continuity of the utility function.

The proof of this result follows the main steps of Arrow and Hahn's proof. First it is proved that the restriction of the preference relation to any closed set $C^+(x)$ can be represented by a utility function defined via a simple formula involving the distance (see Beardon, 1997, Lemma 3). Then, by a suitable extension procedure, a utility function on the whole space is obtained by the utility functions previously constructed (see Beardon, 1997, Theorem 1).

The description of the Euclidean distance approach mentioned above surely suggests some issues. First of all, as pointed out by Bridges and Mehta, "bearing in mind that we expect a consumer to seek consumption bundles that maximize the value of his utility function (subject to budgetary constraints), we would prefer to construct a utility function that is upper, rather than lower, semi-continuous" (see Bridges and Mehta, 1995, pp. 28–29). Moreover, as underlined by Alcantud and Mehta (2009), the extension procedure used by Arrow and Hahn (1971) and Beardon (1997) vitiates the distance approach because it does not allow to have a utility function directly defined in terms of distance. Finally the properties of metric spaces whose distance satisfies (1) and (2) should be carefully analyzed.

Moving from the above considerations, in Section 2 we provide a characterization of the metric spaces considered by Beardon and we show that (1) and (2) imply several topological properties. In fact we prove that a topological space $(X, \tau)$ admits a distance which induces $\tau$ and satisfies (1) and (2) if and only if it is a generalized Peano continuum, that is, a locally compact, locally connected, connected and metrizable topological space. Furthermore we show that such topological spaces always admit a convex metric[2] which satisfies (1) and (2) and preserves the original topology. These facts stress, in particular, that connectedness of the space is implicitly used in Beardon's representation theorem and suggest that such a topological property has an important role in the construction of the utility function through the distance.

[1] Sometimes metric spaces whose closed balls are compact are called *proper metric spaces*.

[2] See Section 2 for the definition of convex metric.