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The almost surely shrinking yolk☆

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ABSTRACT

The yolk, defined by McKelvey as the smallest ball intersecting all median hyperplanes, is a key concept in the Euclidean spatial model of voting. Koehler conjectured that the yolk radius of a random sample from a uniform distribution on a square tends to zero. The following sharper and more general results are proved here: Let the population be a random sample from a probability measure μ on $\mathfrak{R}^m$. Then the yolk of the sample does not necessarily converge to the yolk of μ. However, if μ is strictly centered, i.e. the yolk radius of μ is zero, then the radius of the sample yolk will converge to zero almost surely, and the center of the sample yolk will converge almost surely to the center of the yolk of μ. Moreover, if the yolk radius of μ is nonzero, the sample yolk radius will not converge to zero if μ contains three non-collinear mass points or if somewhere it has density bounded away from zero in some ball of positive volume. All results hold for both odd and even population sizes.

1. Introduction

In the Euclidean spatial model of voting (Downs, 1957; Schofield, 1985, e.g.), voter ideal points are located in $\mathfrak{R}^m$ and voters prefer policies (points) closer to their ideal points under the Euclidean norm. This is perhaps the most widely used voting model, with many applications (e.g. Rapoport and Golan, 1985; Rabinowitz and McDonald, 1986; Shapley and Owen, 1989; Poole and Rosenthal, 1991, 2001; Banks et al., 2002; Schofield, 2004; Banks et al., 2006). Ever since the work of McKelvey (1979) and Schofield (1978) showed that a core would generally not exist, and in fact the space collapses into chaotic cycles, much effort has been made to find a satisfactory solution to the equilibrium problem

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in the spatial model. The yolk, established in Ferejohn et al. (1984) and McKelvey (1986), has emerged as an important solution concept. Defined as the smallest ball intersecting all median hyperplanes, it is the region of policies where a voting game will tend to stabilize. The yolk is also important by virtue of its close relationships to other solution and evaluation concepts, such as the uncovered set (Miller, 1980, 1983; Feld et al., 1987; McKelvey, 1986; Feld et al., 1989), the Pareto set (Feld et al., 1988), the win set (Feld et al., 1987), Shapley–Owen power scores (Feld and Grofman, 1990), epsilon cores (Tullock, 1967; Arrow, 1969; Tovey, 1995), and the finagle point (Wuffle et al., 1989). Several researchers (Feld et al., 1988) have investigated the size of the yolk, and some have speculated that the yolk may tend to be small as the voter population increases. This would be a very desirable property, for if the yolk were very small there would (arguably) be a *de facto* equilibrium even if no single point were undominated. Koehler (1990) combines geometric analysis and simulation to suggest that if voter ideal points are uniformly distributed in a square region of $\mathfrak{R}^2$, the yolk shrinks towards a point as the population grows. We say that the uniform distribution on a square is *strictly centered:* every halfplane that does not contain the center has total probability $< \frac{1}{2}$. In a companion paper (Tovey, 2010) the author applies the Glivenko–Cantelli theory of uniform convergence to prove the following sufficient condition: If n ideal points are sampled at random from a strictly centered, (uniformly) continuous probability measure μ with compact support on $\mathfrak{R}^m$, then the yolk radius converges to 0 and the yolk center converges to the center of the distribution μ, almost surely.

This paper employs more specialized techniques to generalize and sharpen the result just cited. The main results are these: The strict centered condition is, by itself, sufficient for almost sure yolk shrinkage. The same condition comes close to being necessary as well; it is necessary for large classes of probability measures including all discrete distributions in two or more dimensions, and all measures that somewhere put positive continuous density in an open ball. The condition also implies almost sure convergence of the yolk center.

To give these results some intuitive foundation, we use the language of distributional analysis. Suppose the population is a random sample of n points from a probability measure μ on $\mathfrak{R}^m$, inducing empirical measure μ_n. Extend the definition of median hyperplane to distributions: hyperplane h is a median of μ if each halfspace defined by h has measure at least a half, $\mu(h^+) \geq \frac{1}{2}$ and $\mu(h^-) \geq \frac{1}{2}$. The distributional yolk $Y(\mu)$ would then be defined as the smallest ball intersecting all median hyperplanes. One would expect that the yolk of the sample $Y(\mu_n)$ would converge to $Y(\mu)$ as $n \to \infty$, and therefore the natural condition for the radius of $Y(\mu_n)$ to converge to 0 would be that $Y(\mu)$ have zero radius.

However, the yolk of a random sample may fail to converge to the distributional yolk, in both radius and location. An example is given in Section 3. Nonetheless, in this paper we prove that the natural strict centeredness condition, that the radius of $Y(\mu)$ equals zero, is a sufficient and often necessary condition for the sample yolk to shrink to 0 almost surely, in any dimension. In particular, the yolk radius does not converge to zero for any distribution that violates the condition, if somewhere the distribution has density bounded from zero in some open ball, or if it contains three non-collinear mass points, or if it has singular positive density on the surface of a compact manifold with nonzero volume.

These conditions encompass all of the commonly used models of random voter population in the social choice literature, including the standard multivariate normal, uniform distributions on balls, spheres, and hyperrectangles, and all discrete distributions, except for the 1-dimensional discrete. Because many populations of interest are moderate in size, the *rate* of convergence is of concern as well. We obtain an explicit but loose upper bound on the rate of convergence for the uniform distribution on the square.

Building on properties of the yolk established in the literature, our results have similar strong implications regarding the shrinkage and/or non-shrinkage of win sets, the Pareto set, and the uncovered set, and regarding the outcomes of strategic voting under voting agendas, and epsilon cores. The win set $W(x)$ of a point x is the set of points that defeat x by majority vote. Denote the yolk center and radius by c and r, respectively. Let $B(c, k)$ denote the ball of radius k about c; if $k < 0$ then $B(c, k) = \phi$. It is easy to see Miller (2007) that the win set lies between two balls about c, in particular that $W(x) \subseteq B(c, \|x - c\| + 2r)$ and $W(x) \bigcap B(c, \|x - c\| - 2r) = \phi$. Therefore, if $r \to 0$, we have a very good idea as to the location and small size of the win sets. If $y \in W(x) \supset W(y)$, or