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Asymptotics of the minimum manipulating coalition size for positional voting rules under impartial culture behaviour

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ABSTRACT

We consider the problem of manipulation of elections using positional voting rules under impartial culture voter behaviour. We consider both the logical possibility of coalitional manipulation, and the number of voters who must be recruited to form a manipulating coalition. It is shown that the manipulation problem may be well approximated by a very simple linear program in two variables. This permits a comparative analysis of the asymptotic (large-population) manipulability of the various rules. It is seen that the manipulation resistance of positional rules with 5 or 6 (or more) candidates is quite different from the more commonly analyzed three- and four-candidate cases.

1. Introduction

During 1973–75, Gibbard and Satterthwaite published a fundamental impossibility theorem which states that every non-dictatorial social choice function, whose range contains at least three alternatives, at certain profiles can be manipulated by a single individual voter (Gibbard, 1973;

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Satterthwaite, 1975). After that, the natural question arose: if there are no perfect rules, which ones are the best, i.e. least manipulable? To this question there can be no absolute answer — it depends both on the behaviour of the voters, and on the measure used to quantify the term "manipulability".

Among models of voter behaviour, the following two have gained the most attention (Berg and Lepelley, 1994; Kelly, 1993; Saari, 1990). The Impartial Culture (IC) model assumes that voters are independent, and that each voter is equally likely to express any of the possible preference orders among the candidates. The Impartial Anonymous Culture (IAC) model assumes some degree of dependency among the voters. In the present paper, we consider the IC model. This is a very challenging model for social choice rules: since no candidate is inherently favoured by the culture, the voters' collective expressed opinions will create (by chance alone) only slight distinctions between candidates, and it is unlikely that there will be any clear winner. In particular, Condorcet's Paradox occurs more frequently in IC electorates than in more realistically distributed ones (Regenwetter et al., 2006). These features have led to criticism of the IC model as somewhat unrealistic (see Regenwetter et al. (2006)). However, the same features make IC a useful setting in which to study manipulability, since manipulation becomes much easier when the margin of victory is narrow, or when the victor is not a Condorcet winner. Thus, we choose IC in order to focus on situations likely to be manipulable. A necessary caveat to this choice is that the more manipulable parts of another distribution of profiles might not, themselves, resemble the IC distribution.

A realistic study of manipulation would lead us into the theory of political coalitions as canvassed in Riker (1962) and (even more qualitatively) (Riker, 1986). Such an approach soon encompasses considerations (e.g. changing the nature of the issue being voted on) beyond the reach of the mathematical simplicities of social choice theory. Instead, quantitative studies to date have focused on rather stylized notions of manipulability. The most popular measure has been the probability that the votes fall in such a way as to create the (coalitional or individual) "logical possibility of manipulation". This means that some coalition of voters (or individual voter) with incentive to do so can change the election result by voting insincerely. Note in particular that counterthreats are not considered – the manipulator(s) are not opposed by the other, non-strategic voters – and so the existence of a possible manipulation does not imply its presence in a Nash equilibrium in the game-theoretic sense. This model of manipulability has been very widely studied (Baharad and Neeman, 2002; Chamberlin, 1985; Ju, 2005; Kelly, 1993; Kim and Roush, 1996; Lepelley and Mbih, 1987, 1994; Maus et al., 2007; Nitzan, 1985; Pritchard and Slinko, 2006; Pritchard and Wilson, 2007; Saari, 1990). For the case of individual manipulation, some elaborations (the number of individuals who may manipulate, their freedom to do so, and the benefit they derive therefrom) are studied in Aleskerov and Kurbanov (1998) and Smith (1999). The positional (scoring) voting rules have been particular favourites, and significant progress has been made in comparing them. In his seminal paper Saari (1990), Saari showed that in his "geometric" model, Borda's rule is the least manipulable for the three-alternative case in relation to individual manipulation, but that this does not extend to the case of four alternatives.

However, the mere possibility of manipulation sheds little light on the difficulty of carrying it out. For example, how might voters come to discover who can be persuaded to vote insincerely in order to effect a better outcome? Without going into detail concerning such a process, it is clear that the size of the required coalition is of central importance. Intuitively, a situation is more resistant to manipulation if many voters must be recruited to assemble the manipulating coalition, and less resistant if only a few voters are required. In this paper, we consider the probability that a coalition *of at most k voters* can manipulate ($k = 1, 2, \ldots$). Equivalently, we study the probability distribution of the size of the smallest manipulating coalition (a random variable). Similar ideas are explored, in a more limited way, in Pritchard and Slinko (2006) and Pritchard and Wilson (2007).

We use the following notation and assumptions throughout. An election is held to choose one from among m candidates ($m \geq 3$). There are n voters, who hold opinions according to the IC model. That is, each voter is (independently) of one of the $m!$ possible types (preference orders on the candidates), each type being equally likely. The election uses the positional voting rule with score vector $w = (w_1, \ldots, w_m)$, where $1 = w_1 \geq w_2 \geq \cdots \geq w_m = 0$. That is, a vote ranking candidate α in ith place contributes w_i to the score of α, and the candidate with the greatest total score is declared the winner. The possibility of a tie for first place will not be considered in this paper, as Proposition 3 makes it largely irrelevant; it is discussed in detail in Pritchard and Wilson (2007).