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Representation of preferences over a finite scale by a mean operator

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Abstract

Suppose that a decision maker provides a weak order on a given set of alternatives, each alternative being described by a vector of scores, which are given on a finite ordinal scale E. The paper addresses the question of the representation of this weak order by some mean operator, and gives necessary and sufficient conditions for such a representation, with possible shrinking and/or refinement of the scale E.

Keywords: Preference representation; Finite scale; Mean operator; Aggregation of scores; Refinement of scale

JEL Classification: C44; C02

1. Introduction

The representation of preferences is a central topic in decision making and measurement theory. Usually, it amounts to finding a real-valued utility function U such that for any pair of alternatives x, y in some set X of alternatives of interest, $x \succsim y$ iff $U(x) \geq U(y)$. When alternatives are n-dimensional, i.e., $X = \prod_{i=1}^{n} X_i$, a widely studied model is the decomposable model of Krantz et al. (1971), where U has the form:

$$U(x_1, \ldots, x_n) = G(u_1(x_1), \ldots, u_n(x_n)) \tag{1}$$

where the u_i's are real-valued functions. Assuming that $\succsim$ is a weak order on X, it is known that a representation is possible with G being strictly increasing iff $\succsim$ satisfies independence and X is separable (Krantz et al., 1971). A similar condition for G being non decreasing was found by

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Bouyssou and Pirlot, as well as many other results when $\gtrsim$ is not a weak order (Bouyssou and Pirlot, 2004). It is to be noted that in measurement theory, only $\gtrsim$ is supposed to be known, and the marginal utility functions u_i's and G are constructed. In this context, non decreasingness of G appears to be a very natural condition.

Let us address a somewhat different although related problem. Suppose that the set of alternatives is finite and that for each $x \in X$ we know the $u_i(x_i)$'s, in addition to the preference $\gtrsim$. This situation arises in multicriteria evaluation problems, group decision making, etc.: For each alternative x, a *score* is given on some scale E, representing the satisfaction or adequacy of x w.r.t. some criterion or some individual. In real situations, it has to be noted that E is most often a finite scale (e.g., {"bad", "medium", "good"}). These scores play the role of the quantities $u_i(x_i)$. It remains to find a suitable *aggregation function* G to aggregate the marginal scores into a single overall score, which represents the preference $\gtrsim$.

Beside non decreasingness, a natural property for G is internality, which means that the overall score should be comprised between the lowest and the highest marginal scores. Internal and non decreasing aggregation functions are usually called *mean operators*. Hence our main concern will be the representation of preferences by a mean operator.

Few studies have been done in the context of aggregation functions on finite scales, and to the knowledge of the author, almost none deals with preference representation. Conjunctive and disjunctive (hence not internal) aggregation functions on finite scales have been studied in detail by Fodor (2000), and by Mas, Torrens et al. (Mas et al., 1999, 2003). Their result are limited since they consider aggregation functions as mappings from E^n to E, which is obviously very limitative as Examples 1 and 2 will show hereafter. On mean operators on finite scales, there exists a fundamental paper by Ovchinnikov (1996), followed by Marichal and Mesiar (2004), (Marichal et al., 2005), but their point of view is rather different (although complementary of ours) since they are not concerned with preference representation, but with the meaningfulness of means, in the measurement theoretic sense. Concerning this last topic, Rico et al. (2005) have given necessary and sufficient conditions for the representation of preference by a Sugeno integral (Sugeno, 1974; Marichal, 2000b), a particular class of mean operators. Again, their results are limitative since E is considered to be fixed. Therefore, the point of view we adopt here to allow a modification of E in order to have a better ability to represent preferences seems to be original, and opens new horizons.

The paper is organized as follows. We set the framework of the study in Section 2 and state the problem in a clear way. Section 3 gives the main representation result, without considering a refinement of the scale, while Section 4 addresses the case where a refinement is performed. Section 5 indicates possible applications and related works, while Section 6 concludes the paper.

2. Statement of the problem

2.1. Framework and notations

Let E be a finite chain of k elements $e_1 < e_2 < \cdots < e_k$, and consider $A \subseteq E^n$, $n > 1$. Any $a = (a_1, \ldots, a_n) \in A$ is the vector of *scores* of some alternative or object belonging to X (set of potential alternatives), expressing on E some performance, satisfaction, utility, etc. This situation arises, e.g., in multicriteria decision making, group decision making, decision under uncertainty or risk, and when the evaluation of a is given by an interval $[a_1, a_2]$ ($n = 2$) (see Section 6 for a comment