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A new characterization of the path independent choice functions

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Abstract

In this paper we introduce a new axiom for choice functions, equivalent to the path independence axiom. We call it the congruence condition. This axiom allows to construct directly the anti-exchange closure operators and the convex geometries associated to path independent choice functions.

Keywords: Choice function; Convex geometry; Dependence relation; Closure operator; Path independence

JEL classification: D60; D71

1. Introduction

The notion of path independent choice functions (we call them Plott functions) was introduced by Plott (1973). In the literature, we highlight the three following main results. 1) A characterization of the path independence in terms of the heritage and outcast properties (PI$=H\cap O$, Aizerman and Malishevski, 1981), a similar characterization was obtained in Blair et al. (1976). 2) A characterization of Plott functions as joint-extremal choice functions (Aizerman and Malishevski, 1981). 3) A connection of the path independence with convex geometries (Koshevoy, 1999, 2003; Monjardet and Raderanirina, 2001).

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Recall that a choice function f (on a set X) satisfies the *path independence* property (or is a *Plott function*) if

$$f(A\cup B)=f(f(A)\cup B) \qquad \forall A,B\subset X.$$

Note that this property is meaningful for any *operator* on X, that is for any mapping f: $2^X\rightarrow 2^X$. For example, any closure operator satisfies path independence.

Path independence of an operator f implies the following weaker property. Namely, an equality $f(A)=f(A')$ implies $f(A\cup B)=f(A'\cup B)$ for any $B\subset X$. We call this property the *congruence condition*. It is remarkable that, in the case of choice functions, the congruence condition implies path independence (Theorem 1).

The equality $f(A)=f(A\cup b)$ might be thought as a kind of a claim that the item b "depends on" A. Given an operator f we say that B "depends on" A (and denote by $A\vdash_f B$) if $f(A)=f(A\cup B)$, where A, B are subsets of X. We show in Proposition 2 that the relation $\vdash_f$ generated by an operator f is a dependence relation[3] if f satisfies the congruence condition. Note also that any dependence relation $\vdash$ can be generated by means of some closure operator.

The dependence relation $\vdash$, generated by a Plott function, is *framed* in the following sense: for any A there exists the least subset $A'\subset A$ (specifically, $A'=f(A)$) such that $A'\vdash A$. Conversely, for any framed dependence relation $\vdash$ there exists a unique choice function f generating $\vdash$ (by Theorem 1, f is a Plott function). Thus, the set PF(X) of Plott functions is in a natural bijection to the set of framed dependence relations. We show further that the framed dependence relations are in one-to-one correspondence with *anti-exchange* closure operators, or with convex geometries. We come to a natural bijection between the set of Plott functions and the set of *convex geometries*. Moreover, this correspondence is compatible with natural lattice structures on these sets (Theorem 2).

The paper is organized as follows. In Section 2 we introduce the congruence condition for operators and show that it is a consequence of path independence. In Section 3 we show that operators with the congruence condition generate dependence relations. The link between dependence relations and closure operators is described in Section 4. In Section 5 we prove that the congruence condition is equivalent to the path independence property for choice functions. In Section 6, we characterize dependence relations generated by Plott functions. A connection with convex geometries is discussed in Section 7.

2. The congruence condition

In what follows X is a finite set. We use the symbols A, B and so on to denote subsets of X, that is elements of 2^X. An *operator* on X is a mapping f: $2^X\rightarrow 2^X$.

Definition. An operator f satisfies the *path independence* condition if

$$f(A\cup B)=f(f(A)\cup B)$$

for every A and B.

Plott (1973) devised the path independence property for choice functions (a choice function is a *contracting operator*, i.e. $f(A)\subset A$ for any $A\subset X$). However, this condition is meaningful for any operators.

[3]See Section 3 for a definition.