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Stability and bifurcation analysis for the Kaldor–Kalecki model with a discrete delay and a distributed delay

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HIGHLIGHTS

- Propose a Kaldor–Kalecki model with both discrete and distributed delays.
- Discuss the effect of the distributed delay on system dynamics.
- Analyze Hopf bifurcation by multiple scales.

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ABSTRACT

In this paper, a Kaldor–Kalecki model of business cycle with both discrete and distributed delays is considered. With the corresponding characteristic equation analyzed, the local stability of the positive equilibrium is investigated. It is found that there exist Hopf bifurcations when the discrete time delay passes a sequence of critical values. By applying the method of multiple scales, the explicit formulae which determine the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are derived. Finally, numerical simulations are carried out to illustrate our main results.

1. Introduction

Business cycle, named economic cycle, refers to the total alternate expansion and contraction in economic activities. Business cycles in traditional macroeconomics are closely related with investment fluctuations. Examining thoroughly macroeconomic theories of endogenous cycle, we can find that the delay (between an investment decision and the delivery of investment) and the non-linearity in the investment function are two possible explanations of business cycle phenomenon. The former approach can be found in Kalecki's original model [1], which is a differential equation with delay capable of generating periodical oscillation. In this model, a gestation period which is the time between the moment when an investment decision is made and the time of delivering of the finished real investment is assumed to be exist, i.e. there is a time delay after which capital equipment is available for production. The latter approach to explain business cycles appeared in the work of Kaldor [2]. He presented a nonlinear system modeled by a set of ordinary differential equations, where the nonlinearity of investment and saving function lead to limit cycle solution.

Based on the Kalecki's idea about time delay and the Kaldor business cycle model, Krawiec and Szydlowski [3] formulated the Kaldor–Kalecki business cycle model as follows:

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$$\begin{cases} \dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \dfrac{\mathrm{d}K(t)}{\mathrm{d}t} = I(Y(t-\tau), K(t)) - \delta K(t), \end{cases} \tag{1.1}$$

where Y is the gross product, K is the capital stock, α is the adjustment coefficient in the goods market, δ is the depreciation rate of capital stock, $I(Y, K)$ is the investment function, $S(Y, K)$ is the saving function and τ is a gestation of time delay. They showed that for a small time delay parameter the Kaldor–Kalecki model assumes the form of the Lienard equation [4] and investigated the stability of limit cycle [5]. Also, Zhang and Wei [6] investigated global existence of Hopf bifurcation for system (1.1).

Later, Kaddar and Talibi Alaoui introduced the time delay into capital stock and gross product and constructed the Kaldor–Kalecki model of business cycle with time delay in the following form [7]:

$$\begin{cases} \dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \dfrac{\mathrm{d}K(t)}{\mathrm{d}t} = I(Y(t-\tau), K(t-\tau)) - \delta K(t). \end{cases} \tag{1.2}$$

The dynamical behaviors of system (1.2) have been studied extensively in the literature [7–12]. In Ref. [7], Kaddar and Talibi Alaoui investigated the local Hopf bifurcation. In Ref. [8], they established an explicit algorithm for determining the direction of Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions. In Ref. [9], the global existence of periodic solutions was also proved. In Ref. [10], Wu studied simple zero and double zero singularities of system (1.2), got bifurcation diagrams and hence obtained double limit and heteroclinic bifurcations. In Ref. [11], Wu studied zero-Hopf singularity of system (1.2) and obtained its corresponding bifurcation diagrams. In Ref. [12], Wu studied triple zero singularity of system (1.2) and for this singularity derived the normal form on the center manifold.

Note that all the results mentioned above paid attention to the study of Kaldor–Kalecki model with the discrete time delay only. In fact, in order to develop further the essential idea of endogenous business cycle theory, it is more realistic to assume the past influences the present state over an internal of time. Distributed time delay comes naturally and attracts great attention in applications [13–15]. However, to the best of our knowledge, there is no mathematical investigation on the Kaldor–Kalecki model with distributed delay. In the present paper, we shall introduce a distributed delay and modify the Kaldor–Kalecki model as a system of two differential equations with a discrete delay and a distributed delay in the form of

$$\begin{cases} \dfrac{\mathrm{d}Y(t)}{\mathrm{d}t} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \dfrac{\mathrm{d}K(t)}{\mathrm{d}t} = I(Y(t-\tau), \displaystyle\int_{-\infty}^{t} F(t-s)K(s)\mathrm{d}s) - \delta K(t), \end{cases} \tag{1.3}$$

where $F(s)$ is a nonnegative continuous delay kernel defined and integrable on $[0, +\infty)$, which reflects the influence of the past states on the current dynamics. To get a deep and clear understanding of the modified Kaldor–Kalecki model, we consider system (1.3) with the kernel function

$$F(s) = \alpha_1 \mathrm{e}^{-\alpha_1 s}, \quad \alpha_1 > 0.$$

Taking the delay τ as the bifurcation parameter, we investigate the effect of the delay τ on the dynamics of system (1.3) and obtain some novel results.

Business cycle behavior is always characterized by Hopf bifurcation mathematically. An standard approach to Hopf bifurcation analysis of delay differential equations is center manifold reduction [16], which first reconstructs the equation in a Banach space as an ordinary differential equation of abstract operators and then uses the center manifold theory to reduce the dimension of original problems [17]. However, the calculation of the normal form, which is a long and tedious procedure, is the most difficult part in center manifold reduction. In this paper, we compute the normal form on the center manifold by the method of multiple scales [18], which is a global perturbation scheme and has been applied to delay differential equations [19,20].

The rest of our paper is structured as follows. In Section 2, the stability of the positive equilibrium and the occurrence of local Hopf bifurcation are investigated. In Section 3, the direction and stability of the local Hopf bifurcation are established. Some numerical examples are given to illustrate the validity of the main results in Section 4. Finally, we conclude the paper in Section 5.

2. Existence of Hopf bifurcation

In this section, the stability of the positive equilibrium point and the existence of local Hopf bifurcation will be investigated.

As in Ref. [3], we consider some assumptions on the investment and saving functions:

$$I(Y, K) = I(Y) - \beta K,$$