Physica A 460 (2016) 374–383

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Chaos synchronization of fractional chaotic maps based on the stability condition

Guo-Cheng Wu [a,d], Dumitru Baleanu [b,c,*], He-Ping Xie [d], Fu-Lai Chen [e]

[a] Data Recovery Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang, 641100, China
[b] Department of Mathematics, Cankaya University, 06530 Balgat, Ankara, Turkey
[c] Institute of Space Sciences, Magurele–Bucharest, Romania
[d] College of Water Resource and Hydropower, Sichuan University, Chengdu 610065, China
[e] Department of Mathematics, Xiangnan University, Chenzhou 423000, China

HIGHLIGHTS

- Discrete fractional chaotic maps are proposed in the Riemann–Liouville's sense.
- Chaos synchronization of the map is designed and the control parameters are selected according to the stability conditions.
- The feedback control method is extended to discrete fractional equations.

ARTICLE INFO

Article history:
Received 7 October 2015
Received in revised form 28 March 2016
Available online 17 May 2016

Keywords:
Fractional differences
Fractional Lorenz maps
Time scale
Chaos synchronization

ABSTRACT

In the fractional calculus, one of the main challenges is to find suitable models which are properly described by discrete derivatives with memory. Fractional Logistic map and fractional Lorenz maps of Riemann–Liouville type are proposed in this paper. The general chaotic behaviors are investigated in comparison with the Caputo one. Chaos synchronization is designed according to the stability results. The numerical results show the method's effectiveness and fractional chaotic map's potential role for secure communication.

1. Introduction

The fractional calculus as well as its application has gained much attention in recent years. Due to the nice memory effects, the operators can better depict the complicated structures of nonlinear dynamics. Many applications have been suggested in the past decades, such as diffusion in porous media [1], signal processing [2] and control system [3].

Difference equations and discrete maps appeared frequently in discrete natural phenomena. The existing fractional discrete systems generally can be decoupled in two ways: numerical discretization of fractional differential equations and fractionization of difference equations on time scales. The former one is a numerical formula of fractional continuous models and the Grünwald–Letnikov difference (G–L) is often adopted in the numerical treatment. Considering the G–L definition or the Riemann–Liouville derivative (R–L), the obtained discrete formulae are not real discrete systems since the obtained

* Corresponding author at: Department of Mathematics, Cankaya University, 06530 Balgat, Ankara, Turkey.
E-mail addresses: wuguocheng@gmail.com (G.-C. Wu), dumitru@cankaya.edu.tr (D. Baleanu), xiehp@scu.edu.cn (H.-P. Xie), cflmath@163.com (F.-L. Chen).

http://dx.doi.org/10.1016/j.physa.2016.05.045
0378-4371/

solution is defined on the continuous interval and the solutions are still continuous ones. Theoretically, they are not suitable for modeling of discrete events on lattice spaces or discrete time domains, such as image or signal processing, population growth, stock price etc.

Recently, the fractional difference on time scales [4,5] gives a potential tool for discrete fractional modeling. It has a finite fractional difference form which depends on the difference results of all the past statues. This trait can depict the discrete systems' long history effects or long interactions. On the other hand, chaos and chaos synchronization have extensive applications [6–11]. Discrete maps can readily and iteratively generate chaotic signals. So they were paid much attention in both mathematics and other applied science. The logistic map, the standard map et al. have become basic models [12–16]. However, less work has been done in fractional discrete systems which hold complex chaotic dynamics. In Refs. [17,18], the normal delta difference is replaced by the fractional Caputo one. This treatment introduces the discrete memory effects into the chaotic maps. Then, chaos and synchronization of fractional logistic map were reported. The varied fractional orders lead to different chaotic areas so that the chaotic behaviors become more complicated.

The fractional logistic map of the Caputo type was proposed in Ref. [17]. This paper aims to give a new class of discrete models for secure communications. Fractional chaotic maps in the R–L's sense are investigated here. Chaos synchronization is designed according to the stability conditions. The differences are shown and some new chaotic behaviors are reported.

2. Preliminaries

2.1. Definitions

Let us revisit some definitions on time scales $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. The fractional falling function $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ is used in the fractional sum and differences.

Definition 2.1 (*See Ref. [4]*)**.** For u: $\mathbb{N}_a \to \mathbb{R}$ and $0 < \nu$, the definition of the fractional sum is given by

$$\Delta_a^{-\nu} u(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\omega(s))^{(\nu-1)} u(s), \quad t \in \mathbb{N}_{a+\nu},\ \omega(s) = s+1. \tag{1}$$

The sum has a memory effect when the ν is not an integer number. Then, the fractional R–L like difference is developed as well as their properties and the discrete fractional equations initial value problems are discussed in Ref. [4].

Definition 2.2 (*See Ref. [4]*)**.** For $m-1 < \nu \le m$, the R–L fractional difference reads

$$\Delta_a^{\nu} u(t) := \begin{cases} \dfrac{\Delta^m}{\Gamma(m-\nu)} \displaystyle\sum_{s=a}^{t-(m-\nu)} (t-\omega(s))^{(m-\nu-1)} u(s), & t \in \mathbb{N}_{a+m-\nu},\ m-1 < \nu < m, \\ \Delta^m u(t),\ \nu = m. \end{cases} \tag{2}$$

If the delta difference is put into the summation, one can get the Caputo fractional difference [5].

Remark. The fractional sum is given in a finite difference and it is totally different from the normal G–L's one. In addition, the function is defined on $\mathbb{N}_a = \{a, a+1, a+2, \ldots, \}$ which means the solution does not exist out of the set. Finally, the operators are domain shifting ones. In order to let the summation (1) or (2) make sense, the domains should be varied from $\mathbb{N}_a$ to $\mathbb{N}_{a+\nu}$ and $\mathbb{N}_{a+m-\nu}$, respectively.

2.2. Fractional Logistic map of the R–L type

The following fractional R–L difference equation's existence results are given in [4]

$$\Delta_a^{\nu} u(t) = f(u(t^+), t^+), \quad t^+ = t+\nu-1, \qquad \Delta^{\nu-1} u(t)|_{t=a+1-\nu} = \Omega, \quad t \in \mathbb{N}_{a+1-\nu},\ 0 < \nu \le 1. \tag{3}$$

It has a discrete equivalent form

$$u(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)} \Omega + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} (t-\omega(s))^{(\nu-1)} f(u(s^+), s^+), \quad s^+ = s+\nu-1,\ t \in \mathbb{N}_{a+1}. \tag{4}$$

One can solve the fractional difference equation from Eq. (4).

Let us recall the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u, t). \tag{5}$$