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On the fractal distribution of primes and prime-indexed primes by the binary image analysis

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HIGHLIGHTS

- The fractal nature of prime distribution is studied on binary images of primes.
- The dimension of prime distribution is a non-integer value lower than 2.
- The lacunarity of prime distribution of primes and PIPs have been computed.
- The binary image for prime (and PIP) distribution is similar to a Cantor dust.

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ABSTRACT

In this paper, the distribution of primes and prime-indexed primes (PIPs) is studied by mapping primes into a binary image which visualizes the distribution of primes. These images show that the distribution of primes (and PIPs) is similar to a Cantor dust, moreover the self-similarity with respect to the order of PIPs (already proven in Batchko (2014)) can be seen as an invariance of the binary images. The index of primes plays the same role of the scale for fractals, so that with respect to the index the distribution of prime-indexed primes is characterized by the self-similarity alike any other fractal. In particular, in order to single out the scale dependence, the PIPs fractal distribution will be evaluated by limiting to two parameters, fractal dimension (δ) and lacunarity (λ), that are usually used to measure the fractal nature. Because of the invariance of the corresponding binary plots, the fractal dimension and lacunarity of primes distribution are invariant with respect to the index of PIPs.

1. Introduction

In this paper the distribution of primes is considered by focusing on its similarity with a fractal set (Cantor set). Although the distribution of primes has received much attention in classical literature (see e.g. Refs. [1–3]) only recently there have been many attempts to explain the hidden structure of prime distribution by fractality [4–10]. In almost all these papers the Authors give some proofs that the prime distribution is closely related to fractality. In particular the fractal distribution of prime-indexed primes (PIPs) was the subject of Ref. [5], where the Author "reports the empirical observation of fractal structure in the distribution of prime numbers" by showing that "finite-differenced PIP sequences…exhibit quasi-self similar fractality by prime-index order" [5]. In particular, this Author shows that the order of primes plays the same role

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played by the scale for any fractal. In Ref. [6] a binary map is used to represent the distribution of primes in two dimensions. In the same paper the plot of cumulative frequencies of primes is given [6] and it is shown that this graph is similar to a Cantor function thus suggesting that the distribution of primes within naturals might be a fractal-like distribution. The existence of prime reciprocals in the Cantor set is shown in Ref. [7], for a special set of primes that satisfy the equation $2p+1=3^q$. More in general the existence of a hidden structure is one of the main tasks in signal and time series analysis and it can be checked by giving a proof of the power-law ($1/f$-law or long range) dependence. The power-law is a universal property in almost all physical phenomena, like e.g. phase transitions [11], market fluctuations [12], biology [13], and oceanography [14]. The powerlike behavior was also shown in prime distribution [8]. In Ref. [8] it is proven for the first time that "the distribution of prime numbers displays the $1/f$ noise". Although there is no reason for searching a physical meaning in the set of primes, the power-law or long range dependence, given in Ref. [8], is a fundamental step both for discovering the importance of primes in physics and for deepening the research of power-laws in other mathematical structures.

It can be observed that the source for long-range correlation might be linked with the existence of patchiness in primes. The identification of these patches could be the key point for understanding the large scale structure of primes distribution.

Starting from the point of view, that the distribution of primes is similar to the recursive law of a fractal, in this paper we will investigate this property of primes by analyzing some families of prime-indexed primes (for their definition and properties see also Refs. [5,15–17]).

The main idea is to approach the prime distribution as a discrete time series (a dynamical system) and to define the corresponding correlation matrix (based on a Boolean indicator). Then by evaluating the fractal dimension and lacunarity on a 2D image of this Boolean matrix we derive the fractal properties for primes and prime-indexed primes. This correlation matrix, which gives rise to the so-called recurrence- (or dot-) plots, has been widely used to study the complexity of dynamical systems and time-series (see e.g. Refs. [18–20]). Through the recurrence plots we can identify some typical patterns [18–20] in the prime distribution and their fractal nature will be characterized by computing the fractal parameters of these 2D patterns. We will see that the recurrence plots of prime distribution are similar (but not perfectly equivalent) to the Cantor dust. Moreover, similar to any other fractal, the correlation plot of primes distribution can be characterized by the measures of fractality i.e. fractal dimension δ and lacunarity λ. So that we will characterize the primes and PIPs distribution by computing their fractal dimension and lacunarity of their do-plots.

This paper is organized as follows: preliminary remarks on primes and prime-indexed primes are given in Section 2; Section 3 deals with fractal dimension and lacunarity on the correlation plot for the (Boolean) indicator matrix; in Section 4 the results and discussion on fractal analysis of primes and PIPs are given.

2. Preliminary remarks

2.1. Primes and prime-indexed primes

Let $p_i,\ (i \geq 1,\ p_1 = 2)$ be the ith prime, and

$$\mathbb{P} \stackrel{\text{def}}{=} \{p_i : i \in \mathbb{N}\},$$

the set of all primes in $\mathbb{N}$; the counting function, $\pi(x) : \mathbb{R} \to \mathbb{N}$, is defined as

$$\pi(x) = |\mathbb{P}_x|, \qquad \mathbb{P}_x \stackrel{\text{def}}{=} \{p_i \in \mathbb{P} : p_i \leq x\}, \qquad \mathbb{P}_x \subseteq \mathbb{P}. \tag{1}$$

According to Gauss conjecture $\pi(x)$ asymptotically tends to $x/\log x$,

$$\pi(x) \sim x/\log x \tag{2}$$

so that the prime number theorem

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1, \tag{3}$$

holds.

The sequence of prime indexed primes is the "sequence of primes which have a prime index" [17] or equivalently "the subsequence of $\mathbb{P}$ where the index i is itself prime" [15], so that

$$\mathbb{P}^1 \stackrel{\text{def}}{=} \{p_{p_i} : (p_i \in \mathbb{P}) \wedge (p_{p_i} \in \mathbb{P})\}.$$

More sets of indexed primes can be iteratively defined [5,16] by taking prime index subsequences, so that there are infinite sets $\mathbb{P}^k$ of PIPs of order k. For instance it is:

$$p_1 = 2,\ p_2 = 3, \ldots, p_{p_1} = p_2 = 3, \ldots, p_{p_{p_1}} = p_{p_2} = p_3 = 5, \ldots$$

so that

$$\begin{aligned}
\mathbb{P} &= \{2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 19,\ 23, \ldots\} \\
\mathbb{P}^1 &= \{3,\ 5,\ 11,\ 17,\ 31,\ 41,\ 59,\ 67, \ldots\} \\
\mathbb{P}^2 &= \{5,\ 11,\ 31,\ 59, \ldots\} \\
\vdots\ &\ \vdots\ \ldots.
\end{aligned}$$