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Model Lorentz-like equation with continuous spectrum

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HIGHLIGHTS

- Non-analytical dependence of hydrodynamic eigenvalues is demonstrated.
- The problems with extended hydrodynamics for soft interactions are shown.
- The limits if validity of Grad's methods is explicitly stated for such models.

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ABSTRACT

We present a new model of the Lorentz gas kinetic equation for a system where the integral collision operator has a spectrum consisting of a continuous and discrete part. The spectral gap between the two kinds of the spectrum is an adjustable parameter of the model. This allows for the analysis of the existence and property of the hydrodynamical eigenstates and the meaning of the Grad's method of moments for the transition between hard and soft interactions.

1. Introduction

The kinetic Lorentz models [1–3] are simple, solvable examples of the Boltzmann transport equation. Solutions of these models lead to better understanding of various approximations used in the kinetic theory. One of interesting schemes leading to hydrodynamical equations are derived with Grad's method of moments [4,5]. The resulting set of hyperbolic equations forms an extended hydrodynamic approximation [6,7] which in many respects differs from the conventional set of parabolic equations connected with the Chapman–Enskog procedure. The equations of extended hydrodynamics have finite speeds of propagation [5,8] which is an attractive feature of hydrodynamics especially in the relativistic context [9,10]. There are many works concerning the properties of equations of extended hydrodynamics [11,12], but validity of these approximations is still an open problem. Even for the linearised Boltzmann equation the meaning of hydrodynamics for soft interactions when the discrete part is immersed in the continuous part of the spectrum is not understood at all. Only for hard interaction where the spectral gap is clearly defined the meaning of hydrodynamical approximation is well known [13,14]. For such interaction also the Grad's method can be verified [5] and the regions of validity specified.

The situation is much more complicated for the soft interactions when there is no spectral gap and the continuous part of the spectrum is contained in the strip $[-\nu_0, 0]$ with discrete eigenvalues immersed in the continuum. In this paper we consider a model kinetic equation with spectrum of collision operator consisting of the continuous part in the section $[-(1+a), -(1-a)]$ with $0 \leq a \leq 1$ and the eigenvalue $\lambda_0 = 0$. We analyse the dependence of this eigenvalue as a function of the perturbation $i\bar{k}\bar{v}$ (the Fourier transformation of $\bar{v}\bar{\nabla}$) for different values of a. We compare the solution of the

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kinetic equation with solution of the approximate ones derived with Chapman–Enskog and Grad's method of moments. We show that even for such simple model the presence of continuous spectrum for $a = 1$ leads to non-analytical dependence of $\lambda(k)$ on k of the form $\lambda(k) = k^2 \ln(k)$, and such relation cannot be reproduced with a simple set of differential equations of hydrodynamics.

This paper is organised as follows: in Section 2 we define our model and we construct the solution of the kinetic equation for arbitrary initial data. In Section 3 we analyse the fluid dynamical limit and compare with hydrodynamical approximation for $a \neq 1$. Then in Section 4 we introduce the Grad's method of moments and compare the resulting hydrodynamical equations with these derived from Chapman–Enskog and with exact solutions. Finally in the Appendix we solve the model for $a = 1$ and show that in this case there are algebraic tails for long times and the solutions decay like $t^{-\alpha}$.

2. Definition of the model

We consider an extension of the well known Lorentz model assuming a test particle with density $f(x, v, t)$ moving among the uniformly distributed immobile obstacles. We assume that a vector S is associated with each obstacle (we can think of it as of spin or magnetic moment). All vectors S are aligned in one direction which define the z axis in the velocity space. The test particle hitting the obstacle is absorbed with a probability $\sigma(1 + a\frac{\bar{S}\cdot\bar{v}}{|S|\,|v|}) \cdot f(\bar{x}, \bar{v}, t)$ and then emitted in the direction $\bar{v}$ with probability $\sigma(1 - a\frac{\bar{S}\cdot\bar{v}}{|S|\,|v|}) \cdot \int d_3v'(1 + a\frac{\bar{v}'\bar{S}}{|\bar{v}'|\,|\bar{S}|})v_0^{-2}(\delta(|\bar{v}'| - v_0) \cdot \delta(\phi'))f(x, v', t)$. The term $\int d_3v'(1 + a\frac{\bar{v}'\bar{S}}{|\bar{v}'|\,|\bar{S}|})v_0^{-2}(\delta(|\bar{v}'| - v_0) \cdot \delta(\phi'))f(x, v', t)$ is a measure of the number of particles absorbed by the obstacle in point (x, t) which controls the emission of particles with velocity $\bar{v}$. For $a = 0$ this model reduces to the standard Lorentz gas model and leads to the linear Boltzmann like equation of the form:

$$\partial_t f(\bar{x}, \bar{v}, t) + \bar{v}\bar{\nabla}_x f(x, v, t) = L_0 f(\bar{x}, \bar{v}, t) \tag{2.1}$$

where for $f \in L^2(\mathbb{R}^3)$

$$L_0 f(\bar{v}) = \sigma\left(1 + a\frac{v_z}{|\bar{v}|}\right)\int d_3v'\left(1 + a\frac{v'_z}{|\bar{v}|'}\right)f(v')v_0^{-2}\delta(|\bar{v}'| - v_0)\cdot\delta(\phi') - \sigma\left(1 + a\frac{v_z}{|\bar{v}|}\right)f(\bar{v}) \tag{2.2}$$

where $(|\bar{v}|, \phi, \Theta)$ are spherical coordinates of vector $\bar{v}$ and $a \in [0, 1]$ is a parameter. For the most part of the following we put $\sigma = 1$ as it will be needed in the fluid dynamical limits only. We can make this model fully three dimensional assuming uniform distribution in angle ϕ, but as this does not change the calculation and the conclusions, we will assume the flat configuration as above. In this model $|\bar{v}| = v_0$, $\phi = 0$, and we can consider L_0 as an operator acting on the subspace of the $L^2(\mathbb{R}^3)$ space, namely L^2_Θ defined as:

$$L^2_\Theta := \left\{f(\Theta) \in L^2([-1, 1]) : \int_{-1}^{1} d(\cos\Theta)|f(\Theta)|^2 < \infty\right\} \tag{2.3}$$

with inner product defined as:

$$(f, g) = \frac{1}{2}\int_{-1}^{1} d(\cos\Theta) f^*(\Theta)g(\Theta). \tag{2.4}$$

Operator L_0, considered as mapping $L^2_\Theta \to L^2_\Theta$, has the following properties:

1. L_0 is a self-adjoint, bounded and non-positive operator on L^2_Θ,
2. $L_0 = -\nu(\Theta) + \hat{R}$, where $\hat{R}$ is a compact operator and
   $$v(\Theta) = (1 + a\cos\Theta)$$
3. $\lambda_0 = 0$ is an eigenvalue corresponding to the eigenfunction $f = \text{const}$ separated for $a \neq 1$ from the continuous part of the spectrum with a gap equal to $(1 - a)$,
4. The essential spectrum of L_0 is enclosed in the interval $[-(1 + a), -(1 - a)]$.

In order to solve Eq. (2.1)., we apply Fourier transform in $\bar{x}$ and Laplace transform in t and we obtain the following equation:

$$\begin{aligned} z\hat{f}(\bar{k}, \Theta, z) + ikv\cos\Theta f(\bar{k}, \Theta, z) &= f_0(\bar{k}, \Theta) + (1 + a\cos\Theta)P[(1 + a\cos\Theta)f(\bar{k}, \Theta, z)] \\ &\quad - (1 + a\cos\Theta)f(\bar{k}, \Theta, z) \end{aligned} \tag{2.5}$$

where the projector operator P is defined as:

$$Pf(\Theta) = \frac{1}{2}\int_{-1}^{1} d(\cos\Theta)f(\Theta) \tag{2.6}$$

and we have chosen the $\bar{k} \parallel z$ axis.