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Transition from lognormal to χ^2-superstatistics for financial time series

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HIGHLIGHTS

- For the first time a transition from one superstatistics to another is described, as a function of the time scale considered.
- Relevant example system is financial time series of share price returns on various time scales, good quantitative agreement with data.
- New model interpolating between lognormal and chi-square superstatistics is introduced.

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ABSTRACT

Share price returns on different time scales can be well modelled by a superstatistical dynamics. Here we provide an investigation which type of superstatistics is most suitable to properly describe share price dynamics on various time scales. It is shown that while χ^2-superstatistics works well on a time scale of days, on a much smaller time scale of minutes the price changes are better described by lognormal superstatistics. The system dynamics thus exhibits a transition from lognormal to χ^2 superstatistics as a function of time scale. We discuss a more general model interpolating between both statistics which fits the observed data very well. We also present results on correlation functions of the extracted superstatistical volatility parameter, which exhibits exponential decay for returns on large time scales, whereas for returns on small time scales there are long-range correlations and power-law decay.

1. Introduction

Many well established concepts in mathematical finance (such as the Black–Scholes model) are based on the assumption that an index or a stock price follows a geometric Brownian motion, and as consequence the log returns of these processes are Gaussian distributed. But nowadays it is well known that the log returns of realistic stock prices are typically non-Gaussian with fat tails [1–22]. Such behaviour can be well captured by superstatistical models [2–15]. The basic idea of this method borrowed from nonequilibrium statistical mechanics is to regard the time series as a superposition of local Gaussian processes weighted with a process of a slowly changing variance parameter, often called β. This approach has been applied to many areas of complex systems research, including turbulence, high energy scattering processes, heterogeneous nonequilibrium systems, and econophysics (see e.g. Ref. [11] for a short review). In finance early applications of the superstatistics concept were worked out by Duarte Queiros et al. [6,7] and Ausloos et al. [5]. Van der Straeten and Beck [3] analysed daily closing prices of the Dow Jones Industrial Average index (DJI) and the SP 500 index. They verified that both

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Table 1
Decay rates of correlation functions of volatility for shares of different sectors.

Company	Sector	γ	T (Days)	α	T (min)
Alcoa Inc. (AA)	Basic materials	0.115	18	0.094	11
The Coca-Cola Company (KO)	Consumer goods	0.061	15	0.101	13
Bank of America Corporation (BAC)	Financial	0.057	20	0.073	13
Johnson & Johnson (JNJ)	Healthcare	0.041	16	0.102	13
General Electric Company (GE)	Industrial goods	0.068	18	0.062	14
Wal-Mart Stores Inc. (WMT)	Services	0.036	16	0.092	13
Intel Corporation (INTC)	Technology	0.048	17	0.096	13

log-normal superstatistics and χ^2 superstatistics result in good approximations. Biro and Rosenfeld [4] also studied the data sets of the Dow Jones index and verified that the distribution of log returns is well fitted by a Tsallis distribution. Katz and Li Tian [1] showed that the probability distributions of daily leverage returns of 520 North American industrial companies during the 2006–2012 financial crisis comply with the q-Gaussian distribution which can be generated by χ^2 superstatistics. They also verified in Ref. [2] that the Tsallis entropic parameter q obtained by direct fitting to q-Gaussians coincides with the q obtained from the shape parameters of the χ^2 distribution fitted to the histogram of the volatility of the returns. Gerig, Vicente and Fuentes [8] consider a similar model that indicates that the volatility of intraday returns is well described by the χ^2 distribution, see also Ref. [9] for related work in this direction.

In this paper, we will carefully analyse for various data sets of historical share prices which type of superstatistics is best suited to model the dynamics. While Tsallis statistics (=q-statistics) is known to be equivalent to χ^2 superstatistics [13,22], there are other types of superstatistics, such as lognormal superstatistics and inverse χ^2 superstatistics [10], which are known to be different from q-statistics (though all these different statistics generate similar distributions if the variance of the fluctuations in β is small [13]). We show that in our analysis χ^2-superstatistics appears best suitable to describe the daily price changes, whereas on much smaller time scales of minutes lognormal superstatistics seems preferable. We analyse the relevant time scale of the changes in the superstatistical parameter β and present results for the decay of correlations in β. For small return time scales, correlation functions exhibit power law decay and there are long memory effects. In the final section, we develop a synthetic stochastic model that fits the data well. This is kind of a hybrid model interpolating between lognormal and χ^2-superstatistics.

This paper is organized as follows. In Section 2 we look at share price returns on large (daily) time scales. In Section 3 we do a similar analysis on small (minute) time scales. In Section 4 we investigate correlations of the superstatistical volatility parameter on both time scales. In Section 5 the hybrid model is introduced. Our final concluding remarks are given in Section 6.

2. Superstatistics of log-returns of share prices on a large time scale

Non-equilibrium system dynamics can often be regarded as a superposition of a local equilibrium dynamics and a slowly fluctuating process of some variance variable β [13]. These types of 'superstatistical' nonequilibrium models are also useful for financial time series [6,7]. In this article, the empirical data we use as an example is the historical stock prices of Alcoa Inc(AA), which is an American company that engages in the production and management of primary aluminium, fabricated aluminium and alumina. We have looked at shares of many other companies as well (see Table 1), with similar results. Our data set covers the period January 1998–May 2013. We study the log return R_i denoted by

$$R_i = \log\left(\frac{S_{i+1}}{S_i}\right) \tag{1}$$

where $i = 0, 1, 2, \ldots, N$; S_i and S_{i+1} are two successive daily closing prices. We consider the normalized log returns

$$u_i = \frac{R_i - \langle R\rangle}{\sqrt{\langle R^2\rangle - \langle R\rangle^2}} \tag{2}$$

which have been rescaled to have variance 1. The symbol $\langle\cdots\rangle$ denotes the long-time average.

From the simplest superstatistics model point of view, the entire time series of stock prices can be divided into n smaller time slices T. We call T optimal window size. Within each T, the financial volatility β is temporarily constant and the log return of the stock price is Gaussian distributed. β has some probability distribution $f(\beta)$ to take a particular value in a given slice. The conditional probability $p(u|\beta)$ is

$$p(u|\beta) = \sqrt{\frac{\beta}{2\pi}}\exp\left(-\frac{1}{2}\beta u^2\right) \tag{3}$$

and the marginal probability distribution of u for long time observation is the average over local Gaussians weighted with the probability density $f(\beta)$

$$p(u) = \int p(u|\beta)f(\beta)\mathrm{d}\beta. \tag{4}$$