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# One-dimensional hyperbolic transport: Positivity and admissible boundary conditions derived from the wave formulation

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## HIGHLIGHTS

- The Cattaneo equation on the interval can display lack of positivity.
- Proper boundary conditions for hyperbolic transport stem from its wave formulation.
- Physical constraints for the boundary conditions can be derived also from dissipation.

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## ABSTRACT

We consider the one-dimensional Cattaneo equation for transport of scalar fields such as solute concentration and temperature in mass and heat transport problems, respectively. Although the Cattaneo equation admits a stochastic interpretation – at least in the one-dimensional case – negative concentration values can occur in boundary-value problems on a finite interval. This phenomenon stems from the probabilistic nature of this model: the stochastic interpretation provides constraints on the admissible boundary conditions, as can be deduced from the wave formulation here presented. Moreover, as here shown, energetic inequalities and the dissipative nature of the equation provide an alternative way to derive the same constraints on the boundary conditions derived by enforcing positivity. The analysis reported is also extended to transport problems in the presence of a biasing velocity field. Several general conclusions are drawn from this analysis that could be extended to the higher-dimensional case.

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## 1. Introduction

Cattaneo transport equation for scalar fields (matter, heat) stands as a valuable attempt to overcome the intrinsic problems of Fickian transport, namely its infinite velocity of propagation [1] (throughout this article we refer as "Fickian" to the wealth of transport problems where the flux of the transported entity is proportional to the concentration gradient, such as in the Fick's equation for mass transport, or to temperature gradient, such as in the Fourier equation for conductive heat transfer).

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This is achieved by introducing an exponentially decaying memory contribution in the flux–concentration constitutive equation. The Cattaneo model finds important applications in some specific transport problems (such as the phenomenon of the second sound in superfluid helium) [2] (see also the review by Joseph and Preziosi [3,4]), and in a variety of other, more engineering-oriented, studies [5–7]. Some authors have developed an approximate mesoscopic derivation of Cattaneo-like transport equations in the form of a generalized hyperbolic Fokker–Planck equation [8,9], and Cattaneo model rests as a fundamental example of generalized constitutive equation in the theory of extended irreversible thermodynamics [10–12].

Its implications in theoretical physics should not be underestimated, as it represents a valuable contribution for extending transport models in a relativistic (Lorentz-invariant) way [13], and it constitutes the starting point for attempting a stochastic interpretation of the Dirac equation [14,15].

However, its validity has been deeply questioned and criticized for space dimension greater than one, as the associated Green function attains negative values [16], so that the propagation of generic non-negative initial conditions can return negative concentration values.

In point of fact, even this observation can be pushed forward, since there exist one-dimensional problems on the finite interval for which the solution of the Cattaneo equation in the presence of non-negative initial and boundary conditions can become negative. This is shown in Section 2 by means of a simple, closed-form example. The Cattaneo equation on the interval has been studied also in Refs. [17,18], for boundary conditions that do not present problems as it regards positivity.

This result may seem in contradiction with the stochastic derivation of the Cattaneo model starting from a simple stochastic equation driven by dichotomous Poisson noise, developed earlier by S. Goldstein [19] and subsequently re-elaborated by M. Kac in a simple but seminal contribution [20]. This contradiction is however purely apparent and can be resolved by formulating the transport problem in the wave-formalism envisaged by Kac [20] and subsequently elaborated by several other authors [18,21–23], just to cite some relevant contributions in the field. The wave analysis indicates unambiguously that not all of the boundary conditions, that are commonly applied in a Fickian framework, are physically admissible in hyperbolic transport schemes. More precisely, the wave-like nature of the transport models with memory dictates bounds and constraints on the admissible boundary conditions preserving positivity. This is shown in Sections 3 and 4 dealing with the desorption experiment addressed in Section 2. The same constraints derived from positivity emerge from the analysis of the $L^2$-norms, i.e., from energetic conditions, once dissipation is enforced (see Section 5).

In this paper the analysis is also extended to one-dimensional problems in the presence of a deterministic biasing velocity field, deriving constraints on boundary conditions in the classical problem of boundary-layer polarization characterizing transport across permeable or perm-selective membranes (see Section 6). The wave formulation of the constraints to be imposed on the maximum initial flux to ensure non-negativity of the solutions is developed in Section 7. Some general conclusions, oriented towards a rigorous formulation of hyperbolic transport models, are addressed in the concluding Section 8.

## 2. The Cattaneo equation on the unit interval

Consider a transport problem for a conserved scalar field $u(x, t)$ on the spatial interval $x \in [0, L]$ and time $t$, in the absence of source terms. In this case, the balance equation is obviously given by

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}, \tag{1}$$

and assume that the flux $J(x, t)$ is related to $u(x, t)$ via a constitutive equation of Cattaneo type

$$J(x,t) = -k\frac{\partial u(x,t)}{\partial x} - \tau_c \frac{\partial J(x,t)}{\partial t}, \tag{2}$$

where $k$ is the "Fickian" diffusivity (conductivity) and $\tau_c$ the characteristic memory time. By substituting Eq. (2) into Eq. (1) we obtain

$$\frac{\partial u(x,t)}{\partial t} + \tau_c \frac{\partial^2 u(x,t)}{\partial t^2} = k\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{3}$$

namely the Cattaneo equation.

As $\tau_c \to 0$, the Cattaneo equation degenerates into a diffusion equation, while as $\tau_c \to \infty$, keeping $k/\tau_c =$ constant, it reduces to a one-dimensional wave equation. Both these two limit conditions, i.e., the parabolic limit and the pure dissipation-free wave equation, bring physical contradiction inside, namely the infinite velocity of propagation characterizing Fickian transport on one hand, and the occurrence of pure wave-like propagation without dissipation on the other hand, as mentioned by Joseph and Preziosi [3,4].

As initial condition at $t = 0$ let

$$u(x,0) = u_{\text{in}} = \text{constant}, \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0. \tag{4}$$

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