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Superstatistics and the quest of generalized ensembles equivalence in a system with long-range interactions

Nelson A. Alves [a], Rafael B. Frigori [b,*]

[a] Departamento de Física, FFCLRP, Universidade de São Paulo, Avenida Bandeirantes, 3900, 14040-901, Ribeirão Preto, SP, Brazil

[b] Universidade Tecnológica Federal do Paraná, Rua Cristo Rei, 19, 85902-490, Toledo, PR, Brazil

HIGHLIGHTS

- Exact χ^2-superstatistics solution of infinite-range BEC model is provided.
- χ^2-superstatistics is shown to interpolate from canonical to microcanonical ensembles.
- χ^2-superstatistics is shown to be equivalent to EGE solution for BEC model.

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ABSTRACT

The so-called χ^2-superstatistics of Beck and Cohen (BC) is employed to investigate the infinite-range Blume–Capel model, a well-known representative system displaying inequivalence of canonical and microcanonical phase diagrams. While not being restricted to any of those particular thermodynamic limits, our analytical result can smoothly recover both canonical and microcanonical ensemble solutions as its nonextensive parameter q is properly tuned. Additionally, we compare our findings to ones previously obtained from a generalized canonical framework named Extended Gaussian ensemble (EGE). Finally, we show that both EGE and BC solutions are equivalent at the thermodynamic level.

1. Introduction

Superstatistics inception by Beck and Cohen [1] was intended to provide an extension of the standard statistical mechanics formalism into a more general one, focusing on describing out-of-equilibrium systems, which are most likely characterized by spatio-temporal fluctuations of an intensive parameter. Its usual formulation [2–4] employs, as a working hypothesis, the argument that fluctuations evolve on a long-time scale, while the studied system can still be locally decomposed in many small cells (subsystems) obeying the equilibrium statistical mechanics characterized, for instance, by an effective local inverse temperature β. For such systems, not only the temperature environment is considered to be a fluctuating quantity, with probability density $f(\beta)$, but also it may carry a spatial modulation as a classical scalar field.

Despite of some early criticism [5], superstatistics has been growing [1,6–9] as a consistent framework able to provide deep physical insights for a large variety of complex nonequilibrium stationary systems. It is corroborated by the fact that the understanding of BC formulation of superstatistics greatly profits from the perspective of a Bayesian formalism, as exposed by Sattin [10]. For instance, let us suppose one is interested in measurements of the energy E emerging as the outcome of an experiment reproducible as many times as desired. The modeling of this experiment depends on some parameters and

* Corresponding author.
E-mail addresses: alves@ffclrp.usp.br (N.A. Alves), frigori@utfpr.edu.br (R.B. Frigori).

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assumptions related to an underlying stochastic process. So, let β be the parameter that mainly accounts for the dynamics of such experiment, whose due description then implies a proper knowledge of the parameter. Within the Bayesian framework, the knowledge about β is cast in terms of probabilities known as the *prior* distribution $p(\beta)$ and, the *posterior* distribution described by the conditional form $p(\beta|E)$ contains all information about β once the observations on E were given. However, this is clearly an unknown distribution.

In this vein, superstatistics introduces a procedure to obtain marginal probability $p(E)$ from nonequilibrium dynamical processes once given the prior distribution $p(\beta)$ [1,3],

$$p(E) = \int p(E|\beta)p(\beta)\mathrm{d}\beta. \tag{1}$$

This circumvents the "learning" from experiment $p(\beta|E) \sim p(E|\beta)p(\beta)$ [10,11] by assuming that a prior distribution is known, $p(\beta) = f(\beta)$, where $f(\beta)$ physically accounts for the fluctuating parameter β, whose probability distribution function (PDF) is introduced in an *ad hoc* manner and represents our *degree-of-belief*.

The distribution $p(E|\beta)$ in the BC (type-B superstatistics [6]) formulation assumes that the thermodynamical description is statistically performed in the canonical ensemble

$$p(E|\beta) = \frac{\rho(E)\mathrm{e}^{-\beta E}}{Z(\beta)}, \tag{2}$$

where $\rho(E)$ is the density of states and $Z(\beta)$ is the usual normalization constant for a given β. While in the so-called type-A superstatistics formulation $\tilde{p}(E|\beta) = \rho(E)\mathrm{e}^{-\beta E}$ and the normalization of $p(E)$ in Eq. (1) is performed *a posteriori*, i.e. $p(E) = \tilde{p}(E)/Z$, where Z is given by

$$Z = \int_E \int \tilde{p}(E|\beta)p(\beta)\mathrm{d}\beta\, \mathrm{d}E. \tag{3}$$

Then, every time that an *ansatz* may be assumed from the beginning for $p(E|\beta)$ the type-B formulation is considered as more convenient. Different functional forms of $f(\beta)$ have been presented and succeeded in describing many complex physical systems. Among them, we find applications as diverse as in hydrodynamic turbulent flows [12], traffic delays on the British railway network [13], survival-time of cancer patients [14], stock market returns [15] and quark–gluon plasma phenomenology (see Ref. [16], and references therein).

The choice of $f(\beta)$ we will exploit in this study is known as the χ^2-distribution

$$f(\beta) = \frac{1}{\beta_0}\frac{c^c}{\Gamma(c)}\left(\frac{\beta}{\beta_0}\right)^{c-1} \exp\left(-\frac{c\beta}{\beta_0}\right), \tag{4}$$

where constants $\beta_0 > 0$ and, $n = 2c$ is the number of degrees of freedom of the system. In particular, it deserves to be noted that Eq. (4) may partially recover the so-called Tsallis nonextensive statistics when the constant c is formally related to the Tsallis parameter q by $c = 1/(q-1)$. The constant β_0 is related to the average and variance of the spatio-temporal fluctuations of physical quantity β, once $\langle\beta\rangle_f = \int_0^\infty \beta f(\beta)\,\mathrm{d}\beta = \beta_0$ and $\mathrm{Var}(\beta) = \langle\beta^2\rangle_f - \beta_0^2 = \beta_0^2/c$. Still, any coupling of a physical system to finite thermal baths would be expected to be properly described by this somehow interpolating framework, given that β_0 can even be identified with a sharply defined inverse temperature in limit when $\mathrm{Var}(\beta) \to 0$ i.e., if coupled to a thermal reservoir.

This last property would be specially desirable to better understand the thermal behavior of a large set of systems endowed by long-range interactions [17–20] and, whose microcanonical and canonical thermodynamical properties present notable inequivalence. It is broadly accepted that those aspects arise as consequences of the nonconcavity of the entropy, seen as a function of the energy [17,21,22]. Thus, within an interval $(\varepsilon_a, \varepsilon_b)$ of energies where the microcanonical entropy is not a concave function, the microcanonical and the canonical ensembles become nonequivalent. A word of caution may be due here, once we distinguish between the microcanonical entropy $S_\mu(\varepsilon)$ and the canonical one $S_{can}(\beta)$, as the latter is obtained as the Legendre–Fenchel transform of the free energy $\varphi(\beta)$, an operation that always yields a concave function of β.

To circumvent that technical hindrance, Touchette et al. have recently presented ([23], and references therein) a series of methods to analytically calculate entropies that are nonconcave functions of the energy. This can be implemented by some generalized canonical ensembles [24], as the Gaussian Ensemble [25,26] or its extended version (EGE) [27], where the variance of temperature is parameterized (γ) proportionally to the inverse thermal capacity of the heat bath. Therefore, by taking proper limits in this unified approach one can recover usual results in different ensembles [28], even when they are inequivalent in the thermodynamic limit [17].

Thereby, maybe it comes as a startling remark upon aforesaid generalized ensemble approaches, as the χ^2-superstatistics and EGE, that their universal thermodynamic equivalence (in the sense of Costeniuc [24]) may not be ensured right from the beginning. Then, the quest for the existence of equivalent thermodynamic descriptions of some peculiar physical systems studied under different generalized ensembles can only be set by examining their explicit solutions. This is the purpose of our present article, where a χ^2-superstatistics is employed to investigate by explicit calculations the infinite-range Blume–Capel