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Option pricing under deformed Gaussian distributions

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HIGHLIGHTS

- The paper presents and studies a class of deformed geometric Brownian motions.
- Deformed Gaussian distributions are obtained generalizing Tsallis distribution.
- Such deformed processes show an ability in describing fat tails.
- These models describe a complete market: no price of risk is required.
- Implied volatilities for European options are evaluated.

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ABSTRACT

In financial literature many have been the attempts to overcome the option pricing drawbacks that affect the Black and Scholes model. Starting from the Tsallis deformation of the usual exponential function, this paper presents, in a complete market setup, a class of deformed geometric Brownian motions flexible enough to reproduce fat tails and to capture the volatility behavior observed in models that consider both stochastic volatility and jumps.

1. Introduction

In this paper, we present a family of complete market option pricing models with stochastic volatility in which the underlying asset exhibits heavier tails than those of a log-normal distribution. More precisely, the rate of return of this asset follows a diffusion process in the usual form of a geometric Brownian motion but whose source of uncertainty is played by a power-tailed continuous Markov process. Further, the distribution of this process is, at each time t, related to the Tsallis [1] deformation of the exponential function and reduces, as a particular case, to the standard Gaussian distribution.

It is well known that Black and Scholes [2] (B&S thereafter) and Merton [3] proposed a path-breaking model in modern finance when they derived a closed form formula to price European options under the assumption that the dynamics of the underlying asset follows a standard geometric Brownian motion. This pricing expression is a simple pricing tool for both practitioners and researchers.

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If the B&S formula is used to determine the underlying asset implied volatility, it turns out that such volatility is not constant (as instead assumed by the model), being dependent on both the moneyness (i.e. the ratio between the price of the underlying asset and the option strike price) as well as the time to maturity of the option.

The fact that the distribution of the random rate of return of the underlying asset is normal is, unfortunately, a major drawback for the B&S model. Different approaches, that exhibit heavier tails than those obtained with the log-normal distribution, provide a better fit for the observed returns for many equities as well as stock indices (see, for instance, Platen and Rendek [4]). To mention just a few, local volatility models (for instance Derman and Kani [5]), stochastic volatility models (see, amongst others, Heston [6]), and stochastic volatility models with jumps either only in the dynamics of the underlying asset (Bates [7]) or in the dynamics of both underlying asset and its stochastic volatility (Eraker et al. [8], D'Ippoliti et al. [9]).

Any model with more than one source of uncertainty leads to an incomplete market (Bjork [10]) in which, unless a sensible function expressing the market price of risk is exogenously introduced, it would be impossible to price derivatives. This is not the case with the present article that follows the approach of Hobson and Rogers [11]; in their work these authors deal with stochastic volatility in a complete market framework. Borland [12,13], Borland and Bouchaud [14], and Vellekoop and Nieuwenhuis [15] also follow this vein.

In Borland, [12,13], the author proposed a model for stock prices log-returns in the form of

$$\mathrm{d}\ln(S_t) = \mu\,\mathrm{d}t + \sigma_S \mathrm{d}\Omega_t,$$

where μ and σ_S are constants and the source of uncertainty, Ω_t, is no longer a Wiener process as in the B&S model but, rather, a continuous Markov process evolving in time according to the stochastic differential equation (SDE)

$$\mathrm{d}\Omega_t = g_\alpha(t, \Omega_t)\,\mathrm{d}W_t, \tag{1}$$

where $\alpha \leq 1$ is a given parameter, W_t is a Wiener process, $g_\alpha(t, \Omega_t) = f(t, \Omega_t)^{\alpha/2}$, and the probability distribution $f(t, x)$ satisfies the purely sub-diffusive Fokker–Planck (FP) equation

$$\frac{\partial f(t, x)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, x)^{1+\alpha}. \tag{2}$$

It is well known (see e.g. Plastino and Plastino [16], Tsallis and Bukman [17]) that the solution of (2) is in the form of $f(t, x) \propto \exp_\alpha\left(-\frac{1}{2\sigma^2(t)}x^2\right)$, where $\sigma^2(t) \propto t^{2/(2+\alpha)}$ and $\exp_\alpha(x) = (1 + \alpha x)^{1/\alpha}$ is the deformation of the usual exponential function introduced by Tsallis [1] in Statistical Mechanics. As in Hobson and Rogers [11], the Borland approach [12,13] leads to a pricing model that lies between the Heston model, because of a stochastic volatility behavior, and the B&S model, because of its completeness due to the use of the same Wiener process that drives both the price of the underlying asset and its volatility processes. Unfortunately, as observed in Vellekoop and Nieuwenhuis [15], this approach admits arbitrage. These authors, instead, proposed to use the following "deformed" Geometric Brownian Motion

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma_S S_t \mathrm{d}\Omega_t,$$

or equivalently

$$\mathrm{d}\ln(S_t) = \left(\mu - \frac{1}{2}\sigma_S^2 g_\alpha^2(t, \Omega_t)\right)\mathrm{d}t + \sigma_S \mathrm{d}\Omega_t,$$

thereby keeping the original idea to replace the usual Wiener process with a heavier tailed process. However, as all improvements come with a cost, in all those models, unlike B&S and Heston, it is no longer possible to derive closed or semi-closed formulæ for European option prices. Therefore, some numerical methods have to be used.

The above mentioned results is the motivation for this article: we intend to shed some light into how to depart from the standard rate of return process by introducing a broad range of Tsallis deformations. In fact, unlike earlier papers, to derive the "generalized Gaussian" process having, at time t, the Tsallis distribution $f(t, x) \propto \exp_\alpha\left(-\frac{1}{2\sigma^2(t)}x^2\right)$ we do not require $\sigma(t)$ to be an *a priori* specified function. In other words, $f(t, x)$ satisfies a nonlinear Fokker–Planck equation which is time-dependent through the function $\sigma(t)$. A positive consequence of this is that our models are suitable to describe different types of (possibly non-linear) variance changes with time. However, the flexibility in the choice of $\sigma(t)$ requires to guarantee that the process Ω_t is well defined; the proof of this result is one of the main theoretical contribution of the present work.

Another interesting feature we show is that in our framework we can derive an expression that resembles the market price of risk. This formula obviously depends on the choice made for $\sigma(t)$ so that our model reveals to be very flexible in terms of capturing and representing a large class of volatility surfaces. What in incomplete markets is a necessary input, namely some functions describing the market price of risk, becomes in our approach an output that depends on $\sigma(t)$.

Unlike the papers by Borland, [12,13], Borland and Bouchaud [14] and Vellekoop and Nieuwenhuis [15], here we also explicitly derive the dynamics of the variance process that shows a mean-reverting pattern with both long-term mean and speed reversion that are function of time. This result clearly indicates that the deformed Gaussian approach leads to a sensible representation of the volatility process. A detailed analysis of the behavior of log return mean and variance is relegated in Appendix A.