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Shannon entropy, Fisher information and uncertainty relations for log-periodic oscillators

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HIGHLIGHTS

- We calculate the time-dependent Shannon entropy of three log-periodic oscillators.
- We calculate the time-dependent Fisher information (FI) for the same oscillators.
- Relations among the (FI) and the Stam and Cramer–Rao inequalities are discussed.

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ABSTRACT

We calculate the time-dependent Shannon (S_x and S_p) entropy and Fisher (F_x and F_p) information of three log-periodic oscillators. We obtain a general expression for $S_{x,p}$ and $F_{x,p}$ in the state $n = 0$ in terms of ρ, a c-number quantity satisfying a nonlinear differential equation. For two out of three oscillators $S_{x,p}$ and $F_{x,p}$ depend on time, but $S_x + S_p$ and F_xF_p do not. The other oscillator behaves as the time-independent harmonic oscillator where $S_{x,p}$ and $F_{x,p}$ are all constants. Relations among the Fisher information and the Stam and Cramer–Rao inequalities are also discussed.

1. Introduction

Uncertainty relations play an important role in quantum mechanics. The most known is the Heisenberg uncertainty relation [1], introduced in 1927 by W. Heisenberg in terms of the position and momentum variances. This relation imposes a lower limit for the product of uncertainties in position and momentum.

However, several authors have proposed new uncertainty relations beyond that based on standard deviations. In this scenario the information theory takes place and provides new uncertainty relations based on the Shannon entropy and Fisher information.

The Shannon entropy, introduced by Claude E. Shannon in 1948 [2], is used to find fundamental limits on signal processing operations. It has been applied to many areas of Physics [3]. For instance, in Quantum Mechanics it has been recognized as the uncertainty related to the particle position, or in other words, with the degree of localization.

The Shannon entropy in position (S_x) and momentum (S_p) spaces can be used to obtain entropic uncertainty relations, as that derived by Beckner, Bialynicki-Birula, and Mycielski [4], namely:

$$S_x + S_p \geq D\,(1 + \ln \pi)\,, \tag{1}$$

where D is the space dimension. The entropic uncertainty relations are used as alternatives to the Heisenberg uncertainty relation.

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The calculation of the Shannon entropy and entropic uncertainty relations has been carried out for several time-independent potentials. In 1994, Yáñez, Van Assche and Dehesa [5] calculated S_x and S_p of the isotropic harmonic oscillator and the hydrogen atom in D dimensions. The entropies were explicitly obtained in the ground state and in a few low-lying excited states. In 1996, Majerník and Opatrný [6] calculated S_x and S_p for the stationary quantum states of the harmonic oscillator in terms of its energy and determined the corresponding entropic uncertainty relations. They also investigated the time evolution of S_x and S_p in non-stationary states. In 2011, Ghasemi, Hooshmandasl, and Tavassoly [7] calculated S_x and S_p for the quantum states associated with the isotonic oscillator Hamiltonian. They observed that some eigenstates exhibit squeezing effect in S_x. In 2014, Dong et al. [8] obtained the Shannon entropies and standard deviations for a particle in a symmetrical square tangent potential well. The entropy squeezing in position was observed. Other results for different quantum systems can be found in Refs. [9–13].

The Fisher information was introduced by R.A. Fisher in 1925 [14], as a measure of "intrinsic accuracy" in statistical estimation theory. As the Shannon entropy, it can be employed as a quality of an efficient measurement procedure, used to estimate quantum limits and the system disorder. There are several papers reporting on the calculation of the Fisher information for time-independent quantum systems. In 2005, Romera, Sánchez-Moreno and Dehesa [15] calculated the Fisher information of single-particles systems with a central potential. They proposed a new uncertainty relation involving the Fisher information, which is at the same level of the Heisenberg uncertainty relation. In 2007, Patil and Sen [16] calculated the Fisher information for modified isotropic harmonic oscillator and Coulomb potentials. In 2013, Aquino, Flores-Riveros and Rivas-Silva [3] calculated the Fisher information in both position and momentum spaces for a hydrogen atom confined in soft and hard spherical boxes of varying dimension and strength. Further results can be found in Refs. [17–24].

The calculation of the Shannon entropy and the Fisher information for time-dependent quantum systems was reported by Choi et al. in 2011 [25]. This is important because one can verify how fast the loss of information evolves with time. They calculated $F_x(t)$ and $F_p(t)$ for the quantum oscillator with time-dependent frequency and for the strongly pulsating mass system. For the first system, they observed that the Fisher information oscillates with time, while, for the second one, it is highly peaked at points where the mass approaches to zero. In both cases, the Shannon entropy is time-independent.

In 2010, Özeren [26] showed that depending on the choice of mass $(m(t))$ and angular frequency $(\omega(t))$, the classical solutions of the Hamiltonian exhibit either

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2} m(t)\,\omega^2(t)\,x^2, \tag{2}$$

log-periodic or pseudo-log-periodic behavior. Log-periodic oscillations appear in the calculation of the specific heat for systems of non-interacting bosons with multifractal energy spectrum [27] and in the study of the transport properties of a Hamiltonian system with two degrees of freedom [28].

In 2011, Bessa and Guedes [29] used the Lewis and Riesenfeld invariant method [30] and a unitary transformation to obtain the exact Schrödinger wave functions for the time-dependent log-periodic oscillators.

In this work, we calculate the time-dependent variances (Δx and Δp), the Shannon (S_x and S_p) entropy and the Fisher (F_x and F_p) information of three log-periodic oscillators. The Shannon entropy plays the role of a global measure of the spreading of the probability density, while the Fisher information is a local measure since its time behavior is largely affected by local arrangements of the probability distribution. We obtain general expressions for $S_{x,p}$ and $F_{x,p}$ in the state $n = 0$ in terms of ρ, a c-number quantity satisfying a nonlinear differential equation. For one oscillator the Heisenberg uncertainty relation is minimum and $S_{x,p}$ and $F_{x,p}$ are all constants, indicating this oscillator behaves as the time-independent harmonic oscillator. For the other two oscillators Δx, Δp, $S_{x,p}$ and $F_{x,p}$ depend on time. However, for all three oscillators $S_x + S_p$ and F_xF_p are time-independent. These are interesting results, since the Hamiltonians depend on time. The results allow us to write the inequality $F_xF_p \geq 4$ for all the log-periodic oscillators. We also investigate whether the Cramer–Rao and Stam identities are fulfilled. In Section 2, we briefly outline the definitions needed for the calculations. In Section 3, we present and discuss the results. A summary of the work and conclusions are drawn in Section 4.

2. Theory

The Shannon entropies, in one dimension, in the position (S_x) and momentum (S_p) spaces for continuous probability densities, $\varrho(x,t)$ and $\gamma(p,t)$, are, respectively, given by

$$S_x[\varrho(x,t)] = -\int \varrho(x,t)\ln[\varrho(x,t)]\,dx, \tag{3}$$

$$S_p[\gamma(p,t)] = -\int \gamma(p,t)\ln[\gamma(p,t)]\,dp. \tag{4}$$

The Fisher information for position (F_x) and momentum (F_p) spaces is given by

$$F_x = \int \varrho(x,t)\left[\frac{d\varrho(x,t)}{dx}\right]^2 dx, \tag{5}$$