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# Eigenvector perturbations of complex networks

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## HIGHLIGHTS

- We derive a formula to compute eigenvalue perturbation by eigenvector perturbation.
- Two eigenvector perturbation policies and their respective definitions are given.
- We derive the theoretical allowed ranges of two eigenvector perturbation factors.
- The theoretic allowed values of perturbation factors are validated by experiments.
- We seek out eigenvector perturbation forms equivalent to topological perturbations.

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## ABSTRACT

Recently spectral perturbations, involving eigenvalue and eigenvector perturbations, which have attracted more attentions than conventional topological perturbations, are used to analyze and promote the robustness of complex networks. However, to the best of our knowledge, the study of eigenvector perturbation and the equivalence between it and topological perturbation has not been found yet. In this paper, we first deduce the mathematical relationship between eigenvalue perturbation and its corresponding eigenvector perturbation for network reconstructions. Afterwards, two perturbed forms of eigenvector spectrum, global perturbation and local perturbation, are defined, such that we can examine the impacts of eigenvector perturbations on network robustness, and compare those to the impacts of topological perturbations on robustness. Meanwhile, the theoretical ranges of the allowed values of two eigenvector perturbation factors are derived in terms of the accurate reconstruction condition of networks, and validated by experimental simulations. By comparison our finding is that the eigenvector perturbations we define seem equivalent to topological perturbations.



## 1. Introduction

In recent years, there has been a great interest in the study of the robustness or resilience of complex networks, as one of the most important topics in complex networks [1,2]. The robustness properties of complex networks are common analyzing by perturbing the networks, and evaluating by certain measures under perturbations. These perturbations involve not only the structural changes of complex networks but also their spectral changes [3,4]. The structural changes, also called topological perturbations, are the omission or addition of links and/or nodes, or the rewiring of links, or the modification of link weights. The spectral perturbations, including eigenvalue and eigenvector perturbations, of a network mean any form change in the eigenspectra of its adjacency matrix or Laplacian matrix. Although the form of topological

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perturbation is specific and concrete, spectral perturbation has attracted more attentions than topological perturbation, since the eigenspectra of a network contain wealthier characteristic information of this network. For example, the eigenvalue perturbation method, one of spectral perturbations, has been well developed and applied in quantum mechanics [5]. However, to the best of our knowledge, spectral perturbation and the equivalence of spectral and topological perturbations have not been studied intensively so far. In particular, the study of eigenvector perturbation and the equivalence between it and topological perturbation has not been found out yet.

In our earlier work, we have investigated the impact of topological perturbation on eigenvalue spectrum, the impact of eigenvalue perturbation on topological property, and the equivalence between eigenvalue perturbation and topological perturbation [6]. Acting as another spectral perturbation, eigenvector perturbation also plays the same crucial role in spectral perturbations as eigenvalue perturbation [7]. In this paper, we explore the mathematical relation between eigenvalue perturbation and its corresponding eigenvector perturbation, the perturbed forms and perturbation definitions of eigenvector spectrum, and the allowed ranges of perturbation factors, as well as the impacts of eigenvector perturbations on network robustness, and the equivalence between eigenvector perturbation and topological perturbation.

The topological graph of any network $G(V, E)$ consisting of node set $V$ and link set $E$ with $N$ nodes and $L$ links, can be described by an adjacency matrix $A(G) = (a_{ij})_{N\times N}$, a non-negative matrix, whose elements $a_{ij}$ are positive if there is a link going from node $i$ to node $j$ and zero otherwise, with $a_{ii} \equiv 0$. According to the Perron–Frobenius theorem [8], if $A(G)$ is irreducible, its unique largest eigenvalue is real and positive and the components of the corresponding left and right eigenvectors all are positive.

Here assuming that the links in $G(V, E)$ all are undirected and their weights are the same, its adjacency matrix $A(G)$ is a symmetric zero–one matrix, with $a_{ij} = a_{ji} = 1$ if there is a link between node $i$ and node $j$, else $a_{ij} = a_{ji} = 0$. All eigenvalues are real and $A(G)$ possess an eigenvalue decomposition [9], $A(G) = XDX^T$, where $X = [x_1 \quad x_2 \quad \cdots \quad x_N]$is an orthogonal matrix that forms an orthogonal basis, such that $XX^T = X^TX = I_N$, with the real and normalized eigenvectors $x_1, x_2, \ldots, x_N$ of $A(G)$ as the columns of matrix $X$, corresponding to the eigenvalues $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N$ in descending order and the diagonal matrix $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_{N-1}, \lambda_N)$. Here note that the left and right eigenvectors of any eigenvalue in a symmetric matrix are the same.

## 2. Eigenvector perturbation

### 2.1. Application of perturbation theory

Before exploring the definition of eigenvector perturbation, it is necessary to figure out the change in an eigenvalue when its corresponding eigenvector is perturbed. The reason is that there exists the immanent dependence between eigenvector and eigenvalue of a matrix. What is more important, it is indispensable to the subsequent network reconstructions.

Suppose that any graph is modified so that its adjacency matrix $A$ becomes $A + \zeta C$, where $0 < \zeta \ll 1$, and $C$ is a perturbation matrix with the same order of magnitude as $A$ and the finite nonzero entries. Without loss of generality, we may further assume that these modifications are limited to a relatively small number of links or nodes in a large network, then $\zeta \|C\|_2 \ll \|A\|_2$. We believe that this assumption is in accord with most realistic complex networks and systems. Let $\lambda_i$ denote any eigenvalue of $A$ and its corresponding normalized eigenvector be $x_i$. Correspondingly, $\tilde{\lambda}_i$ and $\tilde{x}_i$ denote any eigenvalue and its eigenvector of $A + \zeta C$, respectively. For undirected networks, we apply the general perturbation formulae [10–12]:

$$\tilde{x}_1 = x_1 + \zeta \sum_{k=2}^{N} \frac{x_k^T C x_1}{\lambda_1 - \lambda_k} x_k + \zeta^2 \sum_{m=2}^{N} \left[ \frac{\left(x_1^T C x_1\right)\left(x_m^T C x_1\right)}{\lambda_1 - \lambda_m} - \sum_{k=2}^{N} \frac{\left(x_k^T C x_1\right)\left(x_m^T C x_k\right)}{\lambda_1 - \lambda_k} \right] \frac{x_m}{\lambda_m - \lambda_1} + O\left(\zeta^3\right), \tag{1}$$

and

$$\tilde{\lambda}_1 = \lambda_1 + \zeta x_1^T C x_1 + \zeta^2 \sum_{k=2}^{N} \frac{\left(x_k^T C x_1\right)^2}{\lambda_1 - \lambda_k} + O\left(\zeta^3\right). \tag{2}$$

From formula (1), the change $\Delta x_1$ of eigenvector $x_1$ can be estimated, using a first-order perturbation result, as

$$\Delta x_1 = \tilde{x}_1 - x_1 \approx \zeta \sum_{k=2}^{N} \frac{x_k^T C x_1}{\lambda_1 - \lambda_k} x_k. \tag{3}$$

Similarly, for the other eigenvectors of $A$, i.e., $i \in \{2, 3, \ldots, N\}$, there exist

$$\Delta x_i = \tilde{x}_i - x_i \approx \zeta \sum_{k=1, \lambda_k \neq \lambda_i}^{N} \frac{x_k^T C x_i}{\lambda_i - \lambda_k} x_k. \tag{4}$$

Note that we must exclude the summation terms in the case of an eigenvalue with multiplicities in Eq. (4) [11]. When using the first-order perturbation from formula (2), the change $\Delta\lambda_1$ of the corresponding eigenvalue $\lambda_1$ equals approximately

$$\Delta\lambda_1 = \tilde{\lambda}_1 - \lambda_1 \approx \zeta x_1^T C x_1. \tag{5}$$