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The self consistent expansion applied to the factorial function

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HIGHLIGHTS

- We study the long time mean square displacement (MSD) of a particle swept by a random flow.
- We obtain the MSD for a family of random flows.
- For ordinary turbulence, which is a member of that family we find that the swept particle super diffuses.
- The exponent characterizing the super diffusion is 6/5.
- We explain why it is difficult to distinguish experimentally the slight super diffusion from diffusion.

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ABSTRACT

Most of the interesting systems in statistical physics can be described as nonlinear stochastic field theories. A common feature in the theoretical study of such systems is that ordinary perturbation theory seldom works. On the other hand, there exists a useful tool for the study of systems of that generic nature. That tool, the Self Consistent Expansion (SCE) is technically similar to the ordinary perturbation expansion, in the sense that it is an expansion around a solvable problem. The key point which distinguishes the SCE from an ordinary perturbation expansion, is that the small parameter of the expansion is adjustable and determined inherently by optimization of the expansion. Therefore, it allows the adaptive SCE to remain accurate relative to the inflexible ordinary expansion.

The goal of the present paper is to present the SCE by applying it to a well-known zero dimensional problem. We choose the evaluation of the factorial function, $x!$, as the test case for the SCE, because the Stirling approximation for that function is one of the best known asymptotic expansions, with a very wide use in statistical physics. We show that the SCE approximation holds for small and even negative arguments of the factorial function, where the Stirling expansion fails miserably. It does so without paying any penalty at high values of the argument, where the Stirling formula is excellent. We present numerical as well as analytic SCE approximations of the factorial function.

Recent decades have witnessed a shift of focus from the study of equilibrium and dynamics of phase transitions [1] to the study of non-equilibrium systems. These include a large variety of families of systems including self-organized criticality [2] and phase transitions between non-equilibrium stationary states [3]. Specifically, some examples include various growth models [4,5], front propagation [6–8] and crack propagation [9,10] among others. One natural approach to such systems

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is the use of the renormalization group (RG) which had very successfully explained universality in equilibrium continuous phase transitions and obtained the relevant critical exponents. In spite of its great success, it turns out that the RG method is not always successful.

One important example of the shortcoming of the RG approach is the Kardar–Parisi–Zhang (KPZ) equation [4] where, apart from the one dimensional case [6], the dynamical renormalization group (DRG) has no access to the properties of the strong coupling phase. In addition, a remarkable result by Wiese [11] shows that the drawback of the DRG in the KPZ system is not a product of a low-order calculation, but rather is intrinsic to the method and extends to all orders. The strength of the DRG is that it is based on the ordinary perturbation expansion which is widely known and used. Its disadvantage is that it fails when the system is in the strong coupling regime. A useful tool that overcomes this problem is the SCE. Technically, it is also a perturbation expansion around a simple solvable system. However, this simple system is described by a set of parameters which are optimized by error minimization in order to mimic the actual complex system.

The SCE was originally introduced by Schwartz and Edwards in the study the KPZ problem [12,13]. It worked incredibly well and produced the exact result for one dimension and results very compatible with the results of simulations in higher dimensions. There are many examples in which the DRG fails and the SCE succeeds including KPZ systems with noise that is algebraically correlated in space [14] and in time [15], a family of non-local models such as the non-local KPZ equation [16], the MBE [17] equation etc. The SCE predicts the stretched exponential decay of the KPZ time-dependent structure factor [18], which was later verified in one dimension by numerical integration of the KPZ equation [19]. The SCE was also used for the study of vortex lines in the three dimensional XY model with random phase shifts [20], turbulence [21], wetting and fracture [8,10].

In spite of its success, the use of SCE is not widespread and the main purpose of the present paper is to present it by explaining its philosophy and its implementation. This is done by applying it to the factorial function, which has a well-known effective and widely used asymptotic expansion, namely, the Stirling expansion. Since the factorial function is tabulated and easily accessible numerically, it is rather simple to compare the results of the SCE and of Stirling's formula, with the exact result. We show that the SCE works much better than Stirling's formula at small arguments, without paying any penalty at large arguments.

We begin by describing the SCE technique in general. Consider a complicated system of physical variables ϕ described by some given dimensionless Hamiltonian $\mathcal{H}\{\phi\}$. We would be interested in the free energy $F = -\ln \int e^{-\mathcal{H}\{\phi\}} \mathcal{D}\phi$ and in averages $\langle A \rangle = \frac{\int A\{\phi\} e^{-\mathcal{H}\{\phi\}} \mathcal{D}\phi}{\int e^{-\mathcal{H}\{\phi\}} \mathcal{D}\phi}$, where $\mathcal{D}\phi$ denotes integration over all physical variables. In most case the Hamiltonian can be split naturally to a sum of two terms, $\mathcal{H} \equiv \mathcal{H}_0 + \mathcal{H}_I$, where $\mathcal{H}_0$ is simple enough to obtain the free energy and averages. Ordinary perturbation expansion, is obtained by introducing a notional λ in front of $\mathcal{H}_I$ and expanding the required quantities in λ. This involves calculation of certain averages with respect to $\mathcal{H}_0$, which is possible because of the simplicity of $\mathcal{H}_0$. Eventually, λ is set to unity. The SCE challenges the "natural" split of the Hamiltonian into a "free Hamiltonian" and an "interaction Hamiltonian". This is done by noting the almost trivial fact that the "natural" $\mathcal{H}_0$ is the only simple Hamiltonian which we can work with. There is in fact an infinite family of such Hamiltonians. We denote this family by $\mathcal{H}_0\{\gamma, \phi\}$ where $\{\gamma\}$ denotes a set of parameters for which $\mathcal{H}_0\{\gamma, \phi\}$ is still simple. The "natural" $\mathcal{H}_0$ belongs to that family of Hamiltonians. We can construct a perturbation expansion, similarly to the expansion around the "natural" $\mathcal{H}_0$ for all the required quantities. Thus, $\langle A \rangle$, for example, calculated to a given order in $\mathcal{H}_1\{\gamma, \phi\} = \mathcal{H}\{\phi\} - \mathcal{H}_0\{\gamma, \phi\}$ is a function of $\{\gamma\}$. The SCE is obtained by deriving a set of parameters $\{\gamma\}$ for which $\mathcal{H}_0\{\gamma, \phi\}$ mimics $\mathcal{H}\{\phi\}$. We define "mimics" in a self-consistent way. We choose a set of averages $\left\{A_\psi\right\}$, obtain the expansion of those averages to a given order, n, $\left\{A_\psi^{(n)}\{\gamma\}\right\}$ and require that the lowest order approximations $\left\{A_\psi^{(0)}\{\gamma\}\right\}$ equal $\left\{A_\psi^{(n)}\{\gamma\}\right\}$. This procedure determines the set $\{\gamma\} = \{\gamma^*\}$ and should provide better approximation than the one obtained from the "natural" split. It is clear that the procedure is not unique because the choice of the $\left\{A_\psi\right\}$ involved is not unique. This gives more flexibility which will not be discussed further here.

The technique described above in general terms will now be applied to a specific simple problem of the evaluation of the factorial function,

$$x! = \int_0^\infty dz z^x e^{-z} \quad \text{for } x > -1. \tag{1}$$

Since we would like to work with Gaussian integrals, we change variables in a way that the integration limits are $\pm\infty$. To this end we choose $z = e^y$, which leads to:

$$x! = \int_{-\infty}^\infty dy \exp[(x+1)y - e^y]. \tag{2}$$

The integrand has a maximum at $y_{\max} = \ln(x+1)$. It will prove helpful to shift the maximum of the integrand to zero with another variable change $\eta = y - y_{\max}$ which leads to:

$$x! = (x+1)^{(x+1)} e^{-(x+1)} \int_{-\infty}^\infty e^{-(x+1)(e^\eta - \eta - 1)} d\eta \tag{3}$$