Physica A 463 (2016) 330–344

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A very efficient approach to compute the first-passage probability density function in a time-changed Brownian model: Applications in finance

Luca Vincenzo Ballestra [a], Graziella Pacelli [b,*], Davide Radi [c]

[a] Department of Statistical Sciences, Università Alma Mater Studiorum di Bologna, Bologna, Italy
[b] Department of Management, Università Politecnica delle Marche, Ancona, Italy
[c] School of Economics and Management, LIUC - Università Cattaneo, Castellanza, Italy

HIGHLIGHTS

- A time-changed Brownian model is considered.
- A method to compute the first-passage probability density function is proposed.
- Several applications in finance are presented.
- The proposed method is extremely accurate and fast.
- The proposed approach performs much better than the finite difference method.

ARTICLE INFO

Article history:
Received 5 August 2015
Received in revised form 15 October 2015
Available online 22 July 2016

Keywords:
Time-changed Brownian motion
First-passage probability
Default risk
Option pricing
System of integral equations
Numerical quadrature

ABSTRACT

We propose a numerical method to compute the first-passage probability density function in a time-changed Brownian model. In particular, we derive an integral representation of such a density function in which the integrand functions must be obtained solving a system of Volterra equations of the first kind. In addition, we develop an ad-hoc numerical procedure to regularize and solve this system of integral equations.

The proposed method is tested on three application problems of interest in mathematical finance, namely the calculation of the survival probability of an indebted firm, the pricing of a single-knock-out put option and the pricing of a double-knock-out put option. The results obtained reveal that the novel approach is extremely accurate and fast, and performs significantly better than the finite difference method.

1. Introduction

Time-changed Brownian motions have several applications in finance, biology and physics, see e.g. Refs. [1–9] and references therein, as they allow one to model various kinds of anomalous diffusions that are frequently encountered in these fields.

One of the time changes that is often considered is

$$t \to t^{\alpha}, \quad 0 < \alpha < 1 \tag{1}$$

* Corresponding author.
E-mail addresses: luca.ballestra@unibo.it (L.V. Ballestra), g.pacelli@univpm.it (G. Pacelli), dradi@liuc.it (D. Radi).

http://dx.doi.org/10.1016/j.physa.2016.07.016
0378-4371/

see, for example, Ref. [10]. It is worth observing that the time change (1), despite its simplicity, leads to a volatility structure which is of particular importance in finance and which is considered, for example, in Ref. [5] (see Remark 3.3 therein).

In the present paper, we consider a geometric Brownian motion in which the diffusion term is time changed as in (1) and we propose an efficient approach to compute the first-passage probability density function of such a process through either one or two barrier levels. This function satisfies an initial–boundary value partial differential problem in two independent variables which must be solved by numerical approximation. To this aim, we develop a suitable integral approach that allows us to reduce the partial differential problem to either a single or a system of two Volterra integral equations of the first kind (depending on whether one or two barriers are considered). Then, the integral equations obtained, which involve only one independent variable, are regularized and solved by means of a suitable numerical method based on product integration. The main advantage of such an approach is that the first-passage probability density function can be easily computed using a direct and fast forward recursion.

A geometric Brownian motion with time-change (1) is well-suited for derivative pricing since it ensures the absence of arbitrage opportunities (see, e.g., Refs. [12] and [13]), contrary to what happens for other anomalous diffusion processes such as the fractional Brownian motion (see, e.g., Ref. [11]). Therefore, the method proposed in the present paper is tested on three problems of interest in mathematical finance. Namely, we consider the calculation of the survival probability of an indebted firm, the pricing of single-knock-out put options and the pricing of double-knock-out put options.

Numerical simulations are performed which reveal that the novel algorithm is extremely efficient from the computational standpoint. In fact, in just a few hundredths of a second, it is always possible to obtain relative errors whose order of magnitude is not greater than 10^{-6} and in some cases is also much smaller than that.

To better highlight the performances of the proposed integral method we also present a comparison with another numerical approach. Now, in the literature a variety of numerical algorithms for pricing barrier options or, more in general, for solving partial differential equations of parabolic type, are available (the interested reader can see, for example, Refs. [14–21] and references therein). In the present paper, the proposed integral method is tested against a finite difference scheme which is commonly employed for barrier option pricing (see, e.g., Refs. [22–27]). In particular, as shown by numerical experiments reported in Section 6, the integral method performs significantly better than the finite difference approach.

The remainder of the paper is organized as follows. Section 2 introduces the time-changed Brownian model and the partial differential problem that yields the first-passage probability density function. Section 3 describes the integral formulation of the partial differential problem shown in Section 2. Section 4 performs the numerical approximation of the Volterra integral equations. Section 5 outlines three problems of interest in finance. Section 6 presents and discusses the results of the numerical simulations. Section 7 concludes.

2. Model

Let a stochastic process S follow the time-changed geometric Brownian motion:

$$\mathrm{d}S\left(t\right)=\mu S\left(t\right)\,\mathrm{d}t+\sigma S\left(t\right)\,\mathrm{d}B^{\alpha}\left(t\right),\quad S\left(0\right)=S_0 \tag{2}$$

where μ is a constant drift parameter, σ is a constant volatility parameter and B^{α} is a time-changed Brownian motion with time change as in (1). It is worth noting that if $\alpha = 1$, B^{α} is a standard Wiener process and thus (2) reduces to the well-known geometric Brownian motion on which the famous Black and Scholes model stands, see Ref. [28]. Moreover, the case $0 < \alpha < 1$ represents subdiffusion, while the case $1 < \alpha < 2$ represents superdiffusion, see Ref. [10].

We are concerned about the first-passage of S through two barrier levels S_L and S_U, with $S_L < S(0) < S_U$. Note that we can also consider the case of a single upper barrier by assuming $S_L \to -\infty$ or the case of a single lower barrier by assuming $S_U \to +\infty$.

Let us apply the change of variable

$$X\left(t\right)=\ln\left(S\left(t\right)\right) \tag{3}$$

and let us define

$$x_0=\ln\left(S_0\right). \tag{4}$$

Then, let us consider the first-passage time:

$$\tau=\inf\left\{t\geq 0\mid X\left(t\right)\leq x_L \text{ or } X\left(t\right)\geq x_U\right\} \tag{5}$$

where $x_L = \ln(S_L)$ and $x_U = \ln(S_U)$. That is, τ is the first time at which the process S breaches either the barrier S_L or the barrier S_U.

Moreover, let $f(t, x)$ denote the first-passage probability density function of X, which is defined as follows:

$$f\left(t,x\right)\,\mathrm{d}x=\mathrm{Prob}\left[x\leq X\left(t\right)<x+\mathrm{d}x,\ t\leq\tau\mid X\left(0\right)=x_0\right],\quad x_L\leq x\leq x_U. \tag{6}$$

From well-known results in probability theory, see e.g. Refs. [10,29,30], $f(t, x)$ satisfies for $t \geq 0$ and $x_L \leq x \leq x_U$ the Fokker–Planck equation:

$$\frac{\partial f\left(t,x\right)}{\partial t}-\frac{\alpha}{2}\sigma^2 t^{\alpha-1}\frac{\partial^2 f\left(t,x\right)}{\partial x^2}+\left(\mu-\frac{\alpha}{2}\sigma^2 t^{\alpha-1}\right)\frac{\partial f\left(t,x\right)}{\partial x}=0,\quad x_L<x<x_U \tag{7}$$