Physica A 436 (2015) 686–693

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Beyond the modulational approximation in the wave triplet interaction

P. Iorra, S. Marini, E. Peter, R. Pakter, F.B. Rizzato *

Instituto de Física, Universidade Federal do Rio Grande do Sul, Caixa Postal 15051, 91501-970, Porto Alegre, RS, Brazil

ARTICLE INFO

Article history:
Received 17 May 2015
Available online 27 May 2015

Keywords:
Nonlinear wave dynamics
Plasma waves

ABSTRACT

The present work investigates the breakdown of the traditional modulational approximation in the three wave nonlinear interaction, the wave triplet interaction. A common way to describe the interaction of three high-frequency carriers is to assume that amplitudes and phases are slowly modulated. This is the basis of the modulational approach, which is accurate when the three wave coupling is weak. We examine the types of dynamics arising when the coupling rises from very small to large values. At large values we detect an abrupt transition where the limited amplitude excursions of the modulational regime reach much larger regions of the appropriate configuration space. Extensions to similar cases are also investigated.

1. Introduction

One of the most well accepted and well established models for nonlinear wave interaction is based on the wave triplet concept [1–4]. The wave triplet arises as an entity of physical significance, as one selects the three most prominent modes of an oscillatory system and investigates the coupled dynamics of these selected modes as an isolated subsystem. In general, one mode starts off with a sizeable energetic content, which is then periodically exchanged with the other two modes in a resonant fashion.

Resonant conditions, thus a key ingredient for an active interaction of the modes, are defined as the matching conditions for the high-frequencies of the slowly modulated carriers involved in the interaction. The decay of mode "1" into the other two ("2" and "3"), for instance, is favoured when the resonant condition $\omega_1 = \omega_2 + \omega_3$ among the three high-frequencies is observed [5].

One promptly sees that, in its canonical form, resonant conditions are heavily based on a clear separation of the time scales referring to the high-frequency and the slow modulational dynamics.

There are however a number of cases where such a distinction between the time scales is not so clear. The fact is that although the high-frequency tends to be a fixed parameter for each of the interacting modes, the low-frequency associated to the modulational process grows with the amplitudes or the coupling strength of the involved waves. If at large enough amplitudes or coupling strengths the modulational frequency becomes comparable to the frequency of the carriers, the modulational approximation fails [6,7].

Plasma physics provides a known case where this scenario applies. In a plasma, Langmuir waves, which are modes vibrating with a large frequency called plasma frequency, interact with lower frequency ion-acoustic waves [8,9]. For extremely low amplitudes, wave coupling is small and the modulational frequency is smaller than the low-frequency of

* Corresponding author.
E-mail address: rizzato@if.ufrgs.br (F.B. Rizzato).

http://dx.doi.org/10.1016/j.physa.2015.05.094
0378-4371/

the ion-acoustic carriers. However, as one increases the mode amplitudes, the characteristic frequency of the ion-acoustic modes becomes comparable to the modulational frequency and the slow modulational approach breaks down [10]. Other cases involving different settings [11] point towards the same direction, so a thorough examination of the problem is of relevance.

The modulational description of a wave system is a major achievement that simplifies an otherwise complicated multimode dynamics. The full dynamics is simplified as it is averaged over rapidly varying wave phases of the carriers, thus allowing the interaction to be examined in terms of statistically observable quantities.

In the present case we will focus attention on the problem of the interaction involving three oscillating modes purely on time domain. Spatial dependence will be left aside and reserved for a future investigation. Resonant conditions involving the carrier frequencies shall be enforced, but no assumptions on slow modulational regimes shall be made *a priori*. This way we will be able to construct a full set of equations and examine the transition from a genuine modulational regime up to and beyond the breakdown of modulational approximations.

The paper is organized as follows: in Section 2 we discuss the basic physical model and introduce the corresponding full set of equations we shall be working with; in Section 3 we obtain modulational approximations for the full set in a number of relevant physical settings, compare the approximations with full simulations, and examine what happens beyond the breakdown of the modulation approach. Finally, in Section 4 we draw our conclusions.

2. The physical model

2.1. Full Lagrangian

We start with the Lagrangian for three interacting modes 1, 2, and 3 in the form

$$L = \frac{1}{2}\sum_{j=1}^{3}\left(\dot{x}_j^2 - \omega_j{}^2 x_j^2 - \frac{\varepsilon_2}{2}x_j^4\right) - \varepsilon_1 x_1 x_2 x_3. \tag{1}$$

The natural frequencies of the three modes are denoted by ω_j with $j = 1, 2, 3$. Parameter ε_1 measures the intensity of the triplet coupling, and ε_2, which will be taken as a very small quantity, is introduced as the coefficient of a quartic term which guarantees circumscription of the dynamics to finite regions of the phase-space. The Lagrangian is similar to models used in the analysis of parametric resonances in highly nonlinear systems of three interacting modes [12]. The particular form for the nonlinear potential chosen here allows to access the role of the three wave interaction. To see this we write down the Euler–Lagrange equations for the three involved modes,

$$\ddot{x}_i = -\omega_i^2 x_i - \varepsilon_1 x_j x_k - \varepsilon_2 x_i^3. \tag{2}$$

In this compact notation, all the three indexes i, j, k running from 1 to 3 differ from each other. It becomes clear that the dynamics of mode "i" is driven by the product term $x_j x_k$, the signature of the triplet interaction.

It is seen that we do not introduce either dissipation [13] or broad band effects [14] in our model. Dynamics can be chaotic and incoherent, but if so, exclusively due to the nonlinear effects of the three-degrees-of-freedom. Proper comparison with dissipative chaos and broad band incoherence shall be deferred for a future investigation.

The main issue we intend to analyse here is the validity of the modulational approximation to the full Lagrangian introduced by expression (1). Let us then dedicate the next section to this task in two instances.

3. The modulation approach and its comparison with full simulations

3.1. The modulational Lagrangian when $\omega_1 \sim \omega_2 \sim \omega_3$

The canonical framework amenable to the modulational approach is the one where all modes have carrier frequencies with the same order of magnitude, all of which are much larger than the modulational frequencies of the corresponding modal phases and amplitudes. As mentioned earlier, we will see that this requires sufficiently small values of the triplet coupling coefficient ε_1 for given mode amplitudes.

To implement the idea of slow modulations, we write each mode of the triplet in the form

$$x_i = \sqrt{\frac{2A_i(t)}{\omega_i}}\sin\left(\omega_i t + \phi_i(t)\right), \tag{3}$$

where, in contrast to the uncoupled case where $\varepsilon_1 \to 0$, now the amplitude A_i and phase ϕ_i are assumed to acquire a slow dependence on time satisfying $\dot{A}_i/A_i \sim \dot{\phi}_i/\phi_i \sim \Omega \ll \omega_i$. The frequency Ω will be defined shortly and at this point can be seen as a measure of the slow time scale of the problem.