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Scaling properties for a family of discontinuous mappings

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HIGHLIGHTS

- We study scaling properties of a family of nonlinear discontinuous maps.
- This family is the discontinuous-map representation of well-known nonlinear systems.
- The exponent characterizing the family of maps defines universality classes.

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ABSTRACT

Scaling exponents that describe a transition from integrability to non-integrability in a family of two-dimensional, nonlinear, and discontinuous mappings are obtained. The mapping considered is parameterized by the exponent γ in the action variable. The scaling exponents describing the behavior of the average square action along the chaotic orbits are obtained for different values of γ; therefore classes of universality can be defined. For specific values of γ our mapping acts as the discontinuous-map representation of well-known nonlinear systems, thus making our study broadly applicable. Also, the formalism used is general and the procedure can be extended to characterize many other dynamical systems.

1. Introduction and model

Parametric Hamiltonians $H(K)$ appear frequently in the study of physical problems of interest, both in the classical and quantum domains, where the change of K may represent the effect of an external field (electric field, magnetic flux, gate voltage) or a change of an effective interaction (as in many body systems). In general, when the perturbation is small $H(K)$ can be split in two parts: H_0 representing the unperturbed system, whose dynamics could be integrable or non-integrable, and H_1 representing the perturbation parameterized by K; that is, $H(K) = H_0 + KH_1$.

In particular, for a two-dimensional (2D) classical system given in action–angle variables $H(K)$ takes the form [1]

$$H(K, I_1, I_2, \theta_1, \theta_2) = H_0(I_1, I_2) + KH_1(I_1, I_2, \theta_1, \theta_2). \quad (1)$$

Here, since H_0 is assumed to be integrable, K controls the transition from integrability to non-integrability. In fact, a very useful approach to study the dynamics of $H(K)$ in Eq. (1) is to consider a Poincaré section defined by the plane $I_1 \times \theta_1$

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taking θ_2 as constant (mod 2π). Then, the generic 2D (Poincaré) mapping which qualitatively describes the behavior of $H(K)$ reads [1]

$$M_K : \begin{cases} I_{n+1} = I_n + Kf(\theta_n, I_{n+1}), \\ \theta_{n+1} = [\theta_n + g(I_{n+1}) + Kh(\theta_n, I_{n+1})] \mod (2\pi), \end{cases} \tag{2}$$

where f, g, and h are assumed to be nonlinear functions, n is the nth iteration of the mapping, and the variables I and θ correspond indeed to I_1 and θ_1, respectively.

Mapping M_K is very general, so depending on the choice of the nonlinear functions (f, g, and h) its dynamics may represent a wide range of different physical systems. As examples, we can mention some well known mappings having in common the choice of $f(\theta_n, I_{n+1}) = \sin(\theta_n)$ and $h(\theta_n, I_{n+1}) = 0$: Chirikov's standard map [2,3], $g(I_{n+1}) = I_{n+1}$, also known as Taylor–Chirikov's map; the bouncer model [4], $g(I_{n+1}) = \xi I_{n+1}$; the logistic twist map [5], $g(I_{n+1}) = I_{n+1} + \xi I_{n+1}^2$; the Fermi–Ulam accelerator model [6,7], $g(I_{n+1}) = 2/I_{n+1}$; a generalized Fermi–Ulam accelerator (*FU*) model [8–11],

$$g(I_{n+1}) = \frac{1}{I_{n+1}^{\gamma}} \quad \text{with} \quad 0 < \gamma \le 1; \tag{3}$$

and the hybrid Fermi–Ulam bouncer model [12], $g(I_{n+1}) = 4\xi^2[I_{n+1} - (I_{n+1}^2 - \xi^{-2})^{1/2}]$ if $I_{n+1} > \xi^{-1}$ and $g(I_{n+1}) = 4\xi^2 I_{n+1}$ if $I_{n+1} \le \xi^{-1}$. Even though the functional form of $g(I)$ for the mappings above vary significantly from one map to another, all share a common dynamical feature: the *generic transition to chaos*[1] driven by the parameter K. In fact, the maps above develop two dynamical regimes separated by the critical parameter K_c. When $K < K_c$, the phase space is composed of stochastic motion bounded by invariant tori, known as KAM (Kolmogorov–Arnold–Moser) scenario [1]. At $K = K_c$, the last KAM curve is destroyed and the transition to global stochasticity takes place. Then, for $K > K_c$, I becomes unbounded and increases diffusively.

It is relevant to stress that the generic transition to chaos shortly described above mainly relies on the choice of $f(\theta_n, I_{n+1})$ made for the maps listed above; i.e. $f(\theta_n, I_{n+1}) = \sin(\theta_n)$. As a mater of fact, when choosing $f(\theta_n, I_{n+1})$ to be the discontinuous function

$$f(\theta_n, I_{n+1}) \equiv f(\theta_n) = \sin(\theta_n)\text{sgn}[\cos(\theta_n)], \tag{4}$$

map M_K (with $h(\theta_n, I_{n+1}) = 0$ and $g(I_{n+1}) = I_{n+1}$) also has two different dynamical regimes delimited by the critical value $K_c = 1$ [15], however for $K < K_c$, M_K does not show stability islands. Actually, due to the discontinuities of $f(\theta)$, KAM theorem is not satisfied and map M_K does not develop the KAM scenario. Moreover, for any $K \neq 0$ the dynamics of this *discontinuous map* is diffusive and a single trajectory can explore the entire phase space [15]. Nevertheless, when $K < K_c$ the dynamics is far from being stochastic due to the sticking of trajectories along cantori (fragments of KAM invariant tori). Examples of physical systems described by discontinuous maps are 2D billiard models like the stadium billiard [16,17] and polygonal billiards [18,19].

Even though some scaling properties of discontinuous maps have been recently studied [20,21] there is still a huge gap in the understanding of such maps, as compared to maps developing the KAM scenario. Thus, in order to contribute to fill this gap, in this paper we study some dynamical properties of the following discontinuous map[2]:

$$M_\gamma : \begin{cases} I_{n+1} = \mid I_n - K\sin(\theta_n)\text{sgn}[\cos(\theta_n)] \mid, \\ \theta_{n+1} = \left[\theta_n + \dfrac{1}{I_{n+1}^{\gamma}}\right] \mod (2\pi) \quad \text{with } 0 < \gamma \le 1. \end{cases} \tag{5}$$

Notice that map M_γ is in fact the discontinuous-map version of the *FU* model [8–11] characterized by the function $g(I)$ given in Eq. (3). We consider here only the case $K < 1$. In addition, note that due to parameter γ in (5), map M_γ represents a *family* of discontinuous maps.[3]

Then, in the following section we will explore and characterize some dynamical properties of map M_γ when $K < 1$. In particular we will focus on the behavior of the average square action $\langle I^2 \rangle$ and the average standard deviation of I, that we name here ω, as a function of the nth iteration of the map as well as the parameters K and γ.

[1] The description of the generic transition to chaos, where Chirikov's standard map is used as a paradigm, can be found in well known textbooks [1,13] as well as in recent research papers, see e.g. Ref. [14].

[2] The absolute value in the first equation of (5) is necessary to avoid the fractional powers of negative numbers in the equation for the phase θ.

[3] In fact, the continuous version of map M_γ (i.e. the *FU* model) represents physically relevant systems for specific values of γ: For $\gamma = 1$, both, the Fermi–Ulam accelerator model [6,7] and the ripple billiard model [22,23] are recovered. For $\gamma = 1/2$ some dynamical properties of a time-dependent potential well [24] are retrieved.