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MaxEnt, second variation, and generalized statistics

A. Plastino *, M.C. Rocca

La Plata National University and Argentina's National Research Council, (IFLP-CCT-CONICET)-C. C. 727, 1900 La Plata, Argentina

HIGHLIGHTS

- We revisit the idea of second variation of the entropic functional.
- We apply it to q-statistics.
- We show that only for heavy-tail distributions the existence of a maximum of the entropy is guaranteed.

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ABSTRACT

There are two kinds of Tsallis-probability distributions: heavy tail ones and compact support distributions. We show here, by appeal to functional analysis' tools, that for lower bound Hamiltonians, the second variation's analysis of the entropic functional guarantees that the heavy tail q-distribution constitutes a maximum of Tsallis' entropy. On the other hand, in the compact support instance, a case by case analysis is necessary in order to tackle the issue.

1. Introduction

During more than 25 years, an important topic in statistical mechanics theory revolved around the notion of generalized q-statistics, pioneered by Tsallis [1]. It has been amply demonstrated that, in many occasions, the celebrated Boltzmann–Gibbs logarithmic entropy does not yield a correct description of the system under scrutiny [2]. Other entropic forms, called q-entropies, produce a much better performance [2]. One may cite a large number of such instances. For example, non-ergodic systems exhibiting a complex dynamics [2].

The non-extensive statistical mechanics of Tsallis has been employed to fruitfully discuss phenomena in variegated fields. One may mention, for instance, high-energy physics [3,4], spin-glasses [5], cold atoms in optical lattices [6], trapped ions [7], anomalous diffusion [8,9], dusty plasmas [10], low-dimensional dissipative and conservative maps in dynamical systems [11–13], turbulent flows [14], Levy flights [15], the QCD-based Nambu, Jona, Lasinio model of a many-body field theory [16], etc. Notions related to q-statistical mechanics have been found useful not only in physics but also in chemistry, biology, mathematics, economics, and informatics [17–19].

In this work we revisit the subject by appeal, in a classical MaxEnt phase–space framework, to the second variation of functionals. We find that such analysis guarantees a maximum of the Tsallis entropy only in the case of the heavy tail distributions. Our present treatment makes it advisable, on a more general MaxEnt framework, to always look at the second functional variation. We begin our discussion by remembering the concept of second variation.

* Corresponding author.
E-mail address: plastino@fisica.unlp.edu.ar (A. Plastino).

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2. Second variation of a functional

The essential concept that we need here is that of *increment h of a functional*. Note that the general theory of Variational Calculus has been developed in a Banach Space (BS) [20]. Particularly important BS instantiations are, of course, Hilbert's space and classical phase–space.

The MaxEnt approach in Banach space requires a first variation that should vanish and a second one that ascertains the nature of the pertinent extremum. This second variation is not usually encountered in MaxEnt practice, since one believes that the entropy possesses a global maximum. This second functional variation is the protagonist of the present endeavor. The approach is described in detail, for instance in the canonical book by Shilov [20] (for local minima). It is simply explained.

One needs to evaluate the increment h of a functional F at the point y of the Banach space one is dealing with. One has

$$F(y+h) - F(y) = \delta^1 F(y,h) + \frac{1}{2}\delta^2 F(y,h^2) + \varepsilon(h) \tag{2.1}$$

where

$$\lim_{h\to 0} \frac{\|\varepsilon(h)\|}{\|h\|^2} = 0. \tag{2.2}$$

By definition, $\delta^1 F(y,h)$ is the first variation of F (if it is linear in h). $\delta^2 F(y,h)$ is F's second variation, quadratic in h. If y is an extremum of F then

$$\delta^1 F(y,h) = 0 \tag{2.3}$$

and it is a local minimum if

$$\delta^2 F(y,h) \geq C\|h\|^2 \quad C > 0, \tag{2.4}$$

or a local maximum if

$$\delta^2 F(y,h) \leq C\|h\|^2 \quad C < 0 \tag{2.5}$$

where C is a constant and $\|h\|$ stands for the norm of h. Accordingly, if the functional F has an extremum at the element y (of the Banach space) then Eq. (2.3) is fulfilled, and if there exists such a constant $C > 0$ (respectively: <0) that Eq. (2.4) (respectively: Eq. (2.5)) is true, then the functional F attains at y a minimum (respectively: a maximum).

In phase–space, the object of our present concerns,

$$\|h\|^2 = \int_M h^2 \, \mathrm{d}\mu \tag{2.6}$$

where M is the region of phase–space one is interested in and μ the associated measure-volume for the concomitant space. We start our consideration with reference to the orthodox instance.

3. Motivation

The following considerations should motivate the reader to seriously consider the importance of elementary notions of functional analysis in q-statistics.

3.1. q-Exponentials as linear functionals or distributions

A generalized function (or *distribution*) is a *continuous functional* defined on a space of test-functions [21]. A typical such test space is the so-called $\mathcal{K}$ space of Schwartz, of infinitely differentiable functions with compact support.

One can prove [21] that x_+^α, defined by

$$x_+^\alpha = x^\alpha, \quad \text{for } x > 0,$$
$$x_+^\alpha = 0, \quad \text{for } x \leq 0 \tag{3.1}$$

is a distribution possessing single poles at integers $\alpha = -k$ with residues (at the pole)

$$R = \frac{(-1)^k}{(k-1)!}\delta^{(k-1)}(x), \tag{3.2}$$

with $k = 1, 2, \ldots, n, \ldots$ [21].

A function is a particular instance of a distribution, called *regular* distribution. A singular distribution is that which cannot represented as a function. For example, Dirac's delta is such a singular distribution. Tsallis' q-exponentials e_q, defined as

$$e_q(x) = [1 + (q-1)x]_+^{\frac{1}{1-q}}, \tag{3.3}$$