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Large deviation estimates involving deformed exponential functions

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HIGHLIGHTS

- A large deviation theorem for fat-tailed distributions is presented.
- q-deformed exponential distributions are studied
- The formalism is developed by analogy with the non-deformed case.

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ABSTRACT

We study large deviation properties of probability distributions with either a compact support or a fat tail by comparing them with q-deformed exponential distributions. Our main result is a large deviation property for probability distributions with a fat tail.

1. Introduction

The Law of Large Numbers (LLN) states that the arithmetic mean of i.i.d. variables $X_1, X_2, \ldots, X_n$ converges to the first moment $\mathbb{E}X_k$ of the probability distribution. The Large Deviation Principle (LDP) is the property that the probability that the arithmetic mean has a deviating value is exponentially small in the number of variables n. It is an important assumption for the theorem of Varadhan [1], which deals with the asymptotic evaluation of certain integrals. See also Refs. [2–7].

Varadhan's theorem is a generalization of Laplace's method of evaluating integrals. As such it is highly relevant for the axiomatic formulation of statistical mechanics. The standard reference in this direction is the book of Ellis [2]. A more recent review is found in Ref. [7]. The breakdown of Varadhan's theorem is related with the occurrence of phase transitions in models of statistical physics. It is due to the appearance of strong correlations between the variables X_k. Another reason of failure of Varadhan's theorem can be that the LDP is not satisfied. This is the case for instance when the probability distribution of the variables X_k has a fat tail. It is the latter situation which is considered in the present work.

Mathematicians have studied large deviations in the context of probability distributions with a fat tail starting with the works of Heyde [8,9] and Nagaev [10,11]. See also Refs. [12–19]. The present work starts from the question whether a systematic use of so-called q-deformed exponential functions can make a contribution to this area of research. The q-deformed exponential functions, used in the present work, have been introduced [20] in the context of non-extensive

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statistical physics [21]. See also Refs. [22,23]. Our approach differs from that of Ref. [24] and of Ref. [25] who consider strong correlations in the context of nonextensive statistical mechanics.

The strategy of the paper is to mimic the standard approach, replacing where meaningful the exponential function by a deformed function. We therefore start in the next section by reviewing some standard inequalities. Section 3 gives the definition of q-deformed exponential and logarithmic functions. Section 4 deals with an application of the Markov inequality in the case of distributions with a compact support. The treatment of distributions with a fat tail is more difficult. Before discussing them in Section 6 we first study the q-exponential distributions in Section 5. Section 7 contains a summary and an evaluation of what has been obtained.

2. The standard inequality

The Markov inequality

$$\text{Prob }(X \geq x) \leq \frac{\mathbb{E}X}{x}, \quad x > 0, \tag{1}$$

valid for any random variable X assuming non-negative values, implies that for any random variable X which assumes real values one has

$$\text{Prob }(X \geq x) \leq A(a)\mathrm{e}^{-ax}, \quad a \geq 0. \tag{2}$$

This expression involves the moment generating function

$$A(a) = \mathbb{E}\mathrm{e}^{aX}. \tag{3}$$

Its existence is called Cramér's condition. For a sequence $X_1, X_2, \ldots, X_n$ of i.i.d. variables there follows

$$\text{Prob}\left(\frac{1}{n}\sum_{k=1}^{n} X_k \geq x\right) \leq A^n(a)\mathrm{e}^{-nax}. \tag{4}$$

Introduce a *rate function* $I(x)$ defined by

$$I(x) = \sup_{\theta\geq 0}\{\theta x - \ln A(\theta)\} \leq +\infty. \tag{5}$$

Note that we change notations from a to θ for compatibility with expressions later on. The function $I(x)$ is convex non-decreasing, with $I(0) = 0$ and $\lim_{x\to+\infty} I(x) = +\infty$ (we assume that $A(a)$ is finite for some $a > 0$).

One obtains

$$\text{Prob}\left(\frac{1}{n}\sum_{k=1}^{n} X_k \geq x\right) \leq \mathrm{e}^{-nI(x)}. \tag{6}$$

When $I(x)$ is strictly positive then an outcome larger than x is a large deviation and its probability decays exponentially fast in n.

3. Deformed logarithmic and exponential functions

Fix q satisfying $0 < q < 2,\ q \neq 1$. The q-deformed logarithm is defined by [20,23]

$$\ln_q(u) = \frac{1}{1-q}\left(u^{1-q} - 1\right), \quad u > 0. \tag{7}$$

In the limit $q = 1$ it reduces to the natural logarithm $\ln u$. The inverse function is the q-deformed exponential. It is defined on the whole of the real axis by

$$\exp_q(u) = [1 + (1-q)u]_+^{1/(1-q)} \leq +\infty. \tag{8}$$

Here, $[u]_+$ denotes the positive part of u. Note that $\exp_q(\ln_q(u)) = u$ holds for all $u > 0$. However, $\ln_q(\exp_q(u))$ may differ from u when $\exp_q(u)$ diverges or vanishes.

For further use we mention that

$$\begin{aligned}\exp_q(u)\exp_{2-q}(-u) &= \left(\frac{[1+(1-q)u]_+}{[1+(1-q)u]_+}\right)^{1/(1-q)} \\ &= 1,\end{aligned} \tag{9}$$

whenever $1 + (1-q)u > 0$.