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Asymptotic behavior for a version of directed percolation on the honeycomb lattice

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HIGHLIGHTS

- A version of directed bond percolation on the honeycomb lattice is studied.
- We derive the critical aspect ratio for the percolation in the thermodynamic limit.
- A critical exponent is determined.
- The asymptotic behavior of the percolation near the critical aspect ratio is obtained.
- A special case of our result gives the Domany–Kinzel model on the honeycomb lattice.

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ABSTRACT

We consider a version of directed bond percolation on the honeycomb lattice as a brick lattice such that vertical edges are directed upward with probability y, and horizontal edges are directed rightward with probabilities x and one in alternate rows. Let $\tau(M, N)$ be the probability that there is at least one connected-directed path of occupied edges from $(0, 0)$ to (M, N). For each $x \in (0, 1]$, $y \in (0, 1]$ and aspect ratio $\alpha = M/N$ fixed, we show that there is a critical value $\alpha_c = (1-x+xy)(1+x-xy)/(xy^2)$ such that as $N \to \infty$, $\tau(M, N)$ is 1, 0 and 1/2 for $\alpha > \alpha_c$, $\alpha < \alpha_c$ and $\alpha = \alpha_c$, respectively. We also investigate the rate of convergence of $\tau(M, N)$ and the asymptotic behavior of $\tau(M_N^-, N)$ and $\tau(M_N^+, N)$ where $M_N^-/N \uparrow \alpha_c$ and $M_N^+/N \downarrow \alpha_c$ as $N \uparrow \infty$.

1. Introduction

Directed percolation, or oriented percolation, can be thought of simply as a percolation process on a directed lattice in which connections are allowed only in a preferred direction. It was first studied by Broadbent and Hammersley in 1957 [1] and it has remained to this day as one of the most outstanding interesting problems in probability and statistical mechanics. Furthermore, directed percolation is closely related to the Reggeon field theory in high-energy physics and the Markov processes with branching, recombination and absorption that occur in chemistry and biology [2,3], etc. Various properties, results and conjectures of directed percolation can be found in Refs. [4,5] and the references therein. Series expansions of the percolation probability have been performed on the directed lattices [6–8]. However very little is known in the way of exact solutions for the directed percolation problem.

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Domany and Kinzel [9] defined a solvable version of compact directed percolation on the square lattice in 1981 as follows. For a fixed $p \in (0, 1)$, each vertical bond is directed upward with occupation probability p (independently of the other bonds) and each horizontal bond is directed rightward with occupation probability 1. Furthermore, it is known that the boundary of the Domany–Kinzel model has the same distribution as the one-dimensional last passage percolation model [10]. A three-dimensional version of Domany–Kinzel model with occupation probability 1 along two spatial directions was considered in Ref. [11]. One of the authors (LCC) had collaborated with F.Y. Wu to consider the model on the square lattice with vertical probability p_v, horizontal probabilities 1 and p_h alternatively [12,13]. Recently we investigated the model on the triangular lattice in terms of a square lattice with vertical probability y, horizontal probabilities 1 and x alternatively, and diagonal edges from lower-left to upper-right or from lower-right to upper-left with probability d [14]. The purpose of this letter is to consider such model on the honeycomb lattice. While the critical exponent for these lattices are the same, the critical value and the asymptotic behavior are different and cannot be obtained by a simple mapping. Notice that the model we study is not a compact directed percolation as holes may exist.

The original honeycomb lattice with regular hexagons is shown in Fig. 1(a). However, it is easier to consider the honeycomb lattice as a brick lattice with the long axis of the bricks horizontal as shown in Fig. 1(b),where vertical probability y, and horizontal probabilities 1 and x alternatively are also shown. Therefore, we shall mainly consider the vertices (sites) of the honeycomb lattice located at a two-dimensional rectangular net $\{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \le m \le M \text{ and } 0 \le n \le N\}$. It is clear that the vertex $(2m, 2n)$ in Fig. 1(b) corresponds to the vertex $(\sqrt{3}m, 3n)$ in Fig. 1(a) for the regular honeycomb lattice. Consider the probabilities $x \in (0, 1]$ and $y \in (0, 1]$ throughout this article, and the percolation always starts from the origin $(0, 0)$. We say that the vertex (m, n) is percolating if there is at least one connected-directed path of occupied edges from $(0, 0)$ to (m, n). Given any $\alpha \in \mathbb{R}$, let $N_\alpha = \lfloor \alpha N \rfloor = \sup\{m \in \mathbb{Z}_+ : m \le \alpha N\}$ with $N \in \mathbb{Z}_+$. The corresponding quantity for the regular honeycomb lattice in Fig. 1(a) will be denoted as α', where $N_{\alpha'}$ is defined as the largest possible $m/\sqrt{3}$, which is equal to or smaller than $\alpha' N$ and m is a positive integer. The other quantities below for the regular honeycomb lattice can be similarly defined.

Let us denote $\mathbb{P}$ as the probability distribution of the bond variables, and define a two point correlation function

$$\tau(N_\alpha, N) = \mathbb{P}((N_\alpha, N) \text{ is percolating}).$$

For $\alpha < 1$, only the vertices $(2n-1, 2n)$ and $(2n, 2n+1)$ have non-zero percolation. Namely, $\tau(N_\alpha, N) = 0$ when $\alpha < 1$ and $N - N_\alpha > 1$. As the percolation of the vertex $(2n-1, 2n)$ is the same as that of $(2n, 2n)$, and the percolation of the vertex $(2n, 2n+1)$ is equal to that of $(2n, 2n)$ multiplied by y, we shall only consider the vertex (m, n) with $m \ge n$, i.e., $\alpha \ge 1$, for the honeycomb lattice here. Equivalently, we only consider $\alpha' \ge 1/\sqrt{3}$ for the regular honeycomb lattice.

It is appropriate to define some of the standard critical exponents and to sketch the phenomenological scaling theory of $\tau(N_\alpha, N)$. A critical value of α exists, that is denoted as α_c and will be derived below. For $\alpha < \alpha_c$ and α close to α_c, the scaling theory of critical behavior asserts that the singular part of $\tau(N_\alpha, N)$ varies asymptotically as (cf. Ref. [15])

$$\tau(N_\alpha, N) \approx \exp\left(\frac{-BN}{(\alpha_c - \alpha)^{-\nu}}\right), \tag{1.1}$$

where the notation $f_\alpha(N) \approx g_\alpha(N)$ means $\lim_{N\to\infty} \log f_\alpha(N)/\log g_\alpha(N) = 1$. The critical exponent $\nu \in (0, \infty)$ is a universal constant [16]. The constant B will be derived explicitly below, that does not depend on α but does depend on x and y. Note that there has been no general proof of the existence of the critical exponents. For $\alpha < \alpha_c$, the critical exponent of the correlation length $\nu = 2$ as shown below is the same as what was found in the Domany–Kinzel model [9,17–19].

The main purpose of this article is to find the critical value

$$\alpha_c = \frac{(1 - x + xy)(1 + x - xy)}{xy^2} = \frac{1 - x^2(1-y)^2}{xy^2} \tag{1.2}$$

for the honeycomb lattice, such that

$$\lim_{N\to\infty} \tau(2N_\alpha, 2N) = \begin{cases} 1 & \text{if } \alpha > \alpha_c, \\ 0 & \text{if } \alpha < \alpha_c, \\ \frac{1}{2} & \text{if } \alpha = \alpha_c. \end{cases} \tag{1.3}$$

Large derivation argument and the Berry–Esseen theorem are used to quantify the rate.

The rest of this paper is organized as follows. In Section 2, we state the main results (Theorems 2.1–2.3). In Section 3, we derive the critical value α_c and the variance σ^2. Theorem 2.1 is proved in Section 4 while the proofs of Theorems 2.2 and 2.3 are similar to those given in Ref. [14] and omitted. A conclusion will be given in Section 5.

2. Main results

First we study the rate of convergence of $\tau(2N_\alpha, 2N)$ for a fixed α. Let us define

$$\underline{\alpha} = \frac{1 + \sqrt{(2\alpha_c - 1)^2 - \sigma^2}}{2}, \tag{2.1}$$