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A random geometric graph built on a time-varying Riemannian manifold

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HIGHLIGHTS

- A scale-free network model is proposed, which is a random geometric graph built on a time-varying Riemannian manifold.
- The model gives a geometric realization of connections made preferentially to more popular nodes and to more similar nodes.
- The model is used to physically simulate the increasing and connecting phenomena of a type of cancer cell.

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ABSTRACT

The theory of random geometric graph enables the study of complex networks through geometry. To analyze evolutionary networks, time-varying geometries are needed. Solutions of the generalized hyperbolic geometric flow are such geometries. Here we propose a scale-free network model, which is a random geometric graph on a two dimensional disc. The metric of the disc is a Ricci flat solution of the flow. The model is used to physically simulate the growth and aggregation of a type of cancer cell.

1. Introduction

In graph theory, a random geometric graph (RGG) is a graph drawn on a bounded region [1]. It is generated by placing nodes on the region randomly and uniformly, and connecting two nodes if the distance between them is no more than a given threshold. RGG makes the research of the structures and properties of complex networks using the theories and methods of geometry possible. For example, the RGG built on the hyperbolic geometry is a scale-free network. The scale-free property is a consequence of the exponential expansion of underlying hyperbolic space [2].

From the mathematical perspective, the RGG built on a static geometry is a static network. In order to analyze evolutionary networks, we need time-varying geometries. Pseudo Riemannian manifolds, e.g. de Sitter space, are such geometries, in which one coordinate can be associated with time. In addition, the connection mechanism of RGG could also be generalized to express multifarious relations between nodes. For example, nodes sprinkled on de Sitter space are endowed with causal relationships, which are induced by the metric. Hence a generalized RGG built on de Sitter space can be generated by connecting each pair of nodes from the young node to the old one if they have causal relationship. This evolutionary graph is

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therefore called causal network [3–5]. The exponential expansions of the areas for the underlying space and the future light cones of the nodes cause the scale-free property of the causal network [6,7]. This causal network model has been simplified and generalized to model citation networks [8]. Besides connecting by causal relationships, a scale-invariant RGG built on Minkowski space introduces another geometric connection mechanism describing the asymmetrical and inhomogeneous relationships emergent from the growing specialization of nodes [9].

In many real growing networks, the popularity of nodes is one of the important factors for connections [10,11]. Connections made preferentially to more popular nodes cause the emergence of the scale-free property [12–14]. Besides the popularity, similarity, known as homophily in social science, of nodes is another important factor for connections in social networks [15–18]. Taking the two connection factors into consideration, the static RGG built on hyperbolic geometry has been extended to a growing network model [19,20]. The connection mechanism of the model is that the new node links to existing nodes by the optimization of trade-offs between popularity and similarity. Hence the model is called popularity similarity optimization model(PSO), which is also a generalized RGG growing in a circle. In the polar coordinates, the angle distance of nodes models the node similarity, and the birth time of nodes models the node popularity.

In this paper, we propose a growing undirected network model, which considers the influences of the popularity and the similarity of nodes in connections. The proposed model is a RGG constructed on a time-varying Riemannian manifold. The manifold is a two dimensional disc with a Ricci flat solution of generalized hyperbolic geometric flow(GHGF) as its metric. GHGF is a system of partial differential equations of metric, which is derived from Einstein's equation [21,22]. We find that the increasing and connecting phenomena of human breast cancer MDA-MB-231 cell line [23] can be physically modeled by the proposed model. With different model parameters, we can physically simulate diffuseness for normal cancer cells and aggregation for treated cancer cells.

This report is organized as follows. The model is proposed in Section 2, and is used to simulate some behaviors of cells in Section 3. The conclusion is drawn in Section 4.

2. The model

Consider a 2-dimensional vector space $\mathbb{R}^2$ with coordinates $\{x, y\}$. Let D be a disc in $\mathbb{R}^2$ around the origin. The metric of D is given by

$$ds^2 = a^{2bt}\left(dx^2 + dy^2\right). \tag{1}$$

Metric (1) is a solution of the generalized hyperbolic geometric flow. Assume the radius of D is R_0 at time $t = 0$. We build an undirected RGG on D by following steps.

For each time t between times $t = 0$ and $t = T$,

1. Draw $[ma^{2bt}]$ nodes uniformly and randomly on D;
2. Draw a disc D_i with area $A \in \mathbb{R}^+$ around each new node i and taking i as center;
3. Connect nodes i and j, if $j \in D_i$ or $i \in D_j$;
4. Scale the coordinates of D with scale factor a^b.

The bracket $[\cdot]$ is the rounding function. The disc D_i is called the neighborhood of node i. Scaling in Step 4 means multiplying the areas of D and all discs in D by the factor a^{2b}, and multiplying all node coordinates in D by the factor a^b. The reason for the scaling is due to that the area element $dv = a^{2bt}dxdy$ grows with t. So, if node i is born at time t, then the area of D_i at current time t_c is $Aa^{2b(t_c-t)}$. When $t \to \infty$, the node density converges to a constant

$$\delta = \frac{\sum_0^{t_c}[ma^{2bt}]}{\pi R_0^2 a^{2bt_c}} \approx \frac{m}{2\pi b \log(a) R_0^2}. \tag{2}$$

We can explain the model by the process of blowing up a balloon. Assume D is a small piece on a big balloon, so it can be considered as flat. Draw some discs with the same area A on D and then blow some air into the balloon at each time. Suppose the number of new drawn discs and the area of balloon surface increase exponentially. Then, the areas of the drawn discs increase exponentially, and the areas of the discs drawn later are smaller than those drawn early. We generate a network by treating the centers of the discs as nodes, and connecting two nodes if one in other's disc. It is easy to check that this network is the same as the above modeled network.

The node neighborhood gives a geometric expression of node similarity in the sense of location: if node i belongs to the neighborhood of node j, we say that i is similar to j. The Boolean similarity here is different from the similarity in the PSO model. The nodes of the PSO model are sprinkled on a circle, and the similarity between two nodes is given by the distance between the angular coordinates of the nodes. The neighborhood also gives a geometric expression of node popularity: node i with larger (smaller) D_i has more (less) chances to attract connections. The growing area of the neighborhood expresses the growing popularity of node. The proposed model gives another way to consider the influences of popularity and similarity in connections.