Physica A 432 (2015) 140–155

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Age, Innovations and Time Operator of Networks

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HIGHLIGHTS

- Extension of the Time Operator theory with network evolution.
- Innovations and Age of Markov Networks.
- A new distribution emerging from the innovations of the Barabási–Albert network evolution model.
- Age of Markov and Barabási–Albert Networks expressed in terms of Tsallis Entropy.

ARTICLE INFO

Article history:
Received 28 December 2014
Available online 26 March 2015

Keywords:
Time Operator
Innovation
Internal Age
Tsallis Entropy
Preferential attachment

ABSTRACT

We extend the Time Operator and Age to Network Evolution models. Internal Age formulas and the distribution of innovations are computed for Erdős–Rényi Random Networks, for Markov Networks and Barabási–Albert preferential Attachment Networks. The innovation probabilities are found to be proportional to the quadratic entropy (which coincides with the Tsallis entropy for entropic index $q = 2$) in all Markov networks, as well as in the linear growth mechanism. The distribution of innovations in the Barabási–Albert model is a new probability distribution of the logarithmic type.

1. Introduction

The idea to represent time as an operator goes back to Pauli [1] in the context of Quantum Mechanics where the concept was not correctly defined due to the semi-boundedness of the spectrum of the Hamiltonian Operator. This problem was resolved in the frame of the Liouville–Von Neumann formulation of Quantum Statistical Mechanics [2–7]. Time Operators were constructed for chaotic Dynamical Systems in the Liouville–Koopman statistical formulation [8,9,3,10–15] and for Stochastic processes [16,17], namely Bernoulli processes [18] and Markov chains [19].

The goal of this work is to extend the Time Operator and Age, to three representative classes of evolving networks. More specifically, the Erdős–Rényi Networks, the Markov Networks and the Barabási–Albert Networks. After generalizing and formulating the Time Operator to evolving networks in Section 2, we define the internal Age for any random adjacency matrix in Section 3. We compute the Age formulas for Erdős–Rényi graphs in Section 4. In Section 5 we study the innovations and Age of three representative Markov networks, namely the Ehrenfest Urn Markov chain, the Moran process and Web navigation. Finally, in Section 6 the innovation probability and Age formula are presented in general for a linear growth mechanism where one link is attached to one of the previous links and for the preferential attachment Barabási–Albert model.

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http://dx.doi.org/10.1016/j.physa.2015.03.026
0378-4371/

2. The Time Operator associated with network evolution

Consider an evolving network with adjacency matrix $a(t) = \{a_{\kappa\lambda}(t)\}, \kappa, \lambda = 1, 2, \ldots, N(t)$, where $N = N(t)$ is the number of nodes at (clock) time t. The network evolution is observed by the realization of the $(N(t))^2$ random variables $a_{\kappa\lambda}(t), \kappa, \lambda = 1, 2, \ldots, N(t)$ of the adjacency matrix at each time t.

The Adjacency Matrix elements are assumed to be mutually independent random variables considered at any specific time t:

$$a_{\kappa\lambda}(t) : \Omega \to \{0, 1\}, \quad \kappa, \lambda = 1, 2, \ldots, N, \ t = 1, 2, \ldots. \tag{1}$$

The probability $w_{\kappa\lambda}(t)$ to have a link from node κ to node λ at time t:

$$w_{\kappa\lambda}(t) = Prob\{a_{\kappa\lambda}(t) = 1\} \tag{2}$$

may be interpreted as the "success" probability of the Bernoulli distributed random variables $a_{\kappa\lambda}(t)$. The evolution law of the probabilities $w_{\kappa\lambda}(t), \ t = 1, 2, \ldots$ is specified by the specific model under consideration. The assumption of independence, is found in several network evolution models, like the Erdős–Rényi random graphs, preferential attachment models, percolation [20–22]. Independence gives the possibility to extend the Time Operator theory to network evolution models. The Time Operator associated to network evolution is defined as follows. The sample space Ω is defined by the possible values of the adjacency matrix elements:

$$\Omega = \{\omega = \{\omega_{\kappa\lambda} = 0, \omega_{\kappa\lambda} = 1\}, \quad \kappa, \lambda = 1, 2, \ldots, N\} = \{0, 1\}^{N^2}.$$

The order N of the adjacency matrix is taken to be larger than the largest order $N(t), \ t = 0, 1, 2, \ldots, T$ that may appear during the duration T of observations, so that all evolutionary states are included. The duration T of observations may be any finite number or infinity in case observations proceed in the remote future $T \to \infty$.

The network evolution defines in a natural way a binary partition of the sample space Ω at each stage t as follows:

$$\xi_{\kappa\lambda}(t) = \{\Xi^0_{\kappa\lambda}(t), \Xi^1_{\kappa\lambda}(t)\} \tag{3}$$

where the two cells are: $\Xi^i_{\kappa\lambda}(t) := \{\omega \in \Omega : \alpha_{\kappa\lambda}(t) = i\}, \ i = 0, 1$.

The successive observations of the network up to stage $t = 1, 2, \ldots$ through the successive partitions are incorporated in the σ-algebra generated by the common refinement $\vee_{\kappa,\lambda} \xi_{\kappa\lambda}(t)$:

$$\mathfrak{S}_t = \mathfrak{S}(a_{11}(t), a_{12}(t), \ldots, a_{NN}(t)) = \mathfrak{S}(\vee_{\kappa,\lambda} \xi_{\kappa,\lambda}(t)) = \bigvee_{\tau=1}^{t} \mathfrak{S}(\xi_{\kappa,\lambda}(\tau)). \tag{4}$$

The notation $\mathfrak{S}(.)$ stands for "the σ-algebra generated by" (.).

The sequence of increasing σ-algebras $\mathfrak{S}_t, \ t = 1, 2, \ldots$ is the natural filtration of the process $a_{\kappa\lambda}(t)$:

$$\{\Omega, \emptyset\} = \mathfrak{S}_0 \subseteq \mathfrak{S}_1 \subseteq \mathfrak{S}_2 \subseteq \cdots \subseteq \mathfrak{S}_T = \mathfrak{S} \tag{5}$$

where $\mathfrak{S} = \mathfrak{S}_T$ is the σ-algebra generated by the final observation.

The binary random variables $\alpha_{\kappa\lambda}(t)$ are square integrable, i.e. they live in the Hilbert space $L^2(\Omega, \mathfrak{S}, \mu)$ with the correlation scalar product: $(X, Y) = E[X, Y]$. In order to describe the fluctuations from the equilibrium value, we denote by $\mathcal{H}_0$ the Hilbert space of constant random variables. The projection of any random variable $\mathbf{Z} \in L^2(\Omega, \mathfrak{S}, \mu)$ onto $\mathcal{H}_0$ is the expected value $E[\mathbf{Z}]$. Therefore the fluctuations $\mathbf{Z} - E[\mathbf{Z}]1_\Omega$ live in the Hilbert space $\mathcal{H} = L^2(\Omega, \mathfrak{S}, \mu) \ominus \mathcal{H}_0$. The filtration Eq. (5) defines the corresponding resolution of the identity $\mathcal{H}_t$ of the fluctuation space $\mathcal{H}$ and the associated resolution of the identity $\mathbb{I}_\mathcal{H}$ by the conditional expectation projections $\mathbb{E}_t$ according to the formulas:

$$\mathcal{H}_t = L^2(\Omega, \mathfrak{S}_t, \mu) \ominus \mathcal{H}_0, \quad t = 0, 1, 2, \ldots, T \tag{6}$$

$$\mathbb{E}_t = E[.|\mathfrak{S}_t] : \mathcal{H} \to \mathcal{H}_t, \quad t = 0, 1, 2, \ldots, T. \tag{7}$$

The resolution properties are straightforward by construction:

$$\bigwedge_{t=1,2,\ldots,T} \mathcal{H}_t = \emptyset \tag{8}$$

$$\bigvee_{t=1,2,\ldots,T} \mathcal{H}_t = \mathcal{H} \tag{9}$$

$$\mathcal{H}_{t_1} \subseteq \mathcal{H}_{t_2}, \quad t_1 < t_2 \tag{10}$$

$$\mathbb{E}_0 = \mathbb{O} \tag{11}$$

$$\mathbb{E}_T = \mathbb{I}_\mathcal{H} \tag{12}$$

$$\mathbb{E}_{t_1} \leq \mathbb{E}_{t_2}, \quad t_1 < t_2. \tag{13}$$

The resolution properties (11)–(13) show moreover that the conditional expectations $\mathbb{E}_t, \ t = 0, 1, \ldots, T$ are also the spectral projections of a unique self-adjoint operator. This operator has been traditionally called Time Operator.