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A concavity property for the reciprocal of Fisher information and its consequences on Costa's EPI

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HIGHLIGHTS

- We consider the entropy power of the sum $X + Z_t$, where Z_t is a Gaussian noise.
- If X has a log-concave density. We extend Costa's result on concavity.
- For log-concave densities the third derivative of entropy power has a positive sign.
- Also, the reciprocal of Fisher information of $X + Z_t$ is concave.

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ABSTRACT

We prove that the reciprocal of Fisher information of a log-concave probability density X in $\mathbb{R}^n$ is concave in t with respect to the addition of a Gaussian noise $Z_t = N(0, tI_n)$. As a byproduct of this result we show that the third derivative of the entropy power of a log-concave probability density X in $\mathbb{R}^n$ is nonnegative in t with respect to the addition of a Gaussian noise Z_t. For log-concave densities this improves the well-known Costa's concavity property of the entropy power (Costa, 1985).

1. Introduction

Given a random vector X in $\mathbb{R}^n$, $n \geq 1$ with density $f(x)$, let

$$H(X) = H(f) = -\int_{\mathbb{R}^n} f(x) \log f(x)\, dx \tag{1}$$

denote its entropy functional (or Shannon's entropy). The entropy power introduced by Shannon [1] is defined by

$$N(X) = N(f) = \frac{1}{2\pi e} \exp\left(\frac{2}{n} H(X)\right). \tag{2}$$

The entropy power is built to be linear at Gaussian random vectors. Indeed, let $Z_\sigma = N(0, \sigma I_n)$ denote the n-dimensional Gaussian random vector having mean vector 0 and covariance matrix σI_n, where I_n is the identity matrix. Then $N(Z_\sigma) = \sigma$. Shannon's entropy power inequality (EPI), due to Shannon and Stam [1,2] (cf. also Refs. [3–8] for other proofs and extensions) gives a lower bound on Shannon's entropy power of the sum of independent random variables X, Y in $\mathbb{R}^n$ with densities

$$N(X + Y) \geq N(X) + N(Y), \tag{3}$$

with equality if and only X and Y are Gaussian random vectors with proportional covariance matrices.

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In 1985 Costa [3] proposed a stronger version of EPI (3), valid for the case in which $Y = Z_t$, a Gaussian random vector independent of X. In this case

$$N(X + Z_t) \geq (1 - t)N(X) + tN(X + Z_1), \quad 0 \leq t \leq 1 \tag{4}$$

or, equivalently, $N(X + Z_t)$, is concave in t, i.e.

$$\frac{d^2}{dt^2} N(X + Z_t) \leq 0. \tag{5}$$

Note that equality to zero in (5) holds if and only if X is a Gaussian random variable, $X = N(0, \sigma I_n)$. In this case, considering that Z_σ and Z_t are independent of each other, and Gaussian densities are stable under convolution, $N(Z_\sigma + Z_t) = N(Z_{\sigma+t}) = \sigma + t$, which implies

$$\frac{d^2}{dt^2} N(Z_\sigma + Z_t) = 0. \tag{6}$$

Let us now consider, for a given random vector X in $\mathbb{R}^n$ with smooth density, its Fisher information

$$I(X) = I(f) = \int_{\{f>0\}} \frac{|\nabla f(x)|^2}{f(x)}\, dx. \tag{7}$$

Blachman–Stam inequality [9,10,2] gives a lower bound on the reciprocal of Fisher information of the sum of independent random vectors with (smooth) densities

$$\frac{1}{I(X + Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}, \tag{8}$$

still with equality if and only X and Y are Gaussian random vectors with proportional covariance matrices.

In analogy with the definition of entropy power, let us introduce the (normalized) reciprocal of Fisher information

$$\tilde{I}(X) = \frac{n}{I(X)}. \tag{9}$$

By construction, since $I(Z_\sigma) = n/\sigma$, $\tilde{I}(\cdot)$ is linear at Gaussian random vectors, with $\tilde{I}(Z_\sigma) = \sigma$. Moreover, in terms of $\tilde{I}$, Blachman–Stam inequality reads

$$\tilde{I}(X + Y) \geq \tilde{I}(X) + \tilde{I}(Y). \tag{10}$$

Therefore, both the entropy power (2) and the reciprocal of Fisher information $\tilde{I}$, as given by (9), share common properties when evaluated on Gaussian random vectors and on sums of independent random vectors.

By pushing further this analogy, in agreement with Costa's result on entropy power, we will prove that the quantity $\tilde{I}(X + Z_t)$ satisfies the analogous of inequality (4), i.e.

$$\tilde{I}(X + Z_t) \geq (1 - t)\tilde{I}(X) + t\tilde{I}(X + Z_1), \quad 0 \leq t \leq 1 \tag{11}$$

or, equivalently

$$\frac{d^2}{dt^2} \tilde{I}(X + Z_t) \leq 0. \tag{12}$$

Unlike Costa's result, the proof of (12) is restricted to log-concave random vectors. Similarly to (5), equality to zero in (12) holds if and only if X is a Gaussian random vector, $X = N(0, \sigma I_n)$.

The estimates obtained in the proof of (12) can be fruitfully employed to study the third derivative of $N(X + Z_t)$. The surprising result is that, at least for log-concave probability densities, the third derivative has a sign, and

$$\frac{d^3}{dt^3} N(X + Z_t) \geq 0. \tag{13}$$

Once again, equality to zero in (13) holds if and only if X is a Gaussian random variable, $X = N(0, \sigma I_n)$. Considering that

$$\frac{d}{dt} N(X + Z_t) \geq 0,$$

the new inequality (13) seems to indicate that the subsequent derivatives of $N(X + Z_t)$ alternate in sign, even if a proof of this seems prohibitive.