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One-dimensional lattices topologically equivalent to two-dimensional lattices within the context of the lattice gas model

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ABSTRACT

Continuum partial differential equations are obtained from a set of discrete stochastic evolution equations of both non-Markovian and Markovian processes and applied to the diffusion within the context of the lattice gas model. A procedure allowing to construct one-dimensional lattices that are topologically equivalent to two-dimensional lattices is described in detail in the case of a rectangular lattice. This example shows the general features that possess the procedure and extensions are also suggested in order to provide a wider insight in the present approach.

1. Introduction

During many decades a lot of work was devoted to develop different approaches and techniques that allow to find the evolution equations of the dynamical variables that describe the problem at hand. Within the context of what could be coined Discrete Stochastic Evolution Equations (DSEE), some papers dealing with examples of evolution equations were written considering only one-dimensional lattices that evolve according to the stochastic lattice gas model [1]. Some examples, showing the versatility of the DSEE approach, can be found in Refs. [2–7]. Particularly, in Ref. [7], a topological theorem was proved that states that every lattice is topologically equivalent to a one-dimensional one. The theorem was included in Ref. [7] in order to justify the study of only one-dimensional problems. In the present paper the explicit construction of a one-dimensional lattice that is topologically equivalent to a two-dimensional lattice will be considered in detail, within the context of the lattice gas model, in order to provide the procedures needed in such an approach. Even when the present approach can be used to describe non-Markovian evolution equations, in this paper, only examples whose evolution rules

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are Markovian will be considered for the sake of simplicity in the presentation. Moreover, only rectangular lattices will be studied, but other interesting lattices like the triangular, and many others, could be included within this context without a considerable effort. As is well known, lots of variants and approaches can be introduced in the highly versatile lattice gas model, and some of them can be found in Refs. [8–13].

The paper is organized as follows. In Section 2 the basic definitions necessary for the *mise en scéne* of the model and notations are introduced for the case of non-Markovian evolution in a d-dimensional lattice and formulas and procedures are introduced. In Section 3, an example that shows not only the way and steps needed to construct the one-dimensional lattice that is topologically equivalent to a two-dimensional one, but how to get rid of some "border anomalies" introducing some periodic boundary conditions according to the lattice at hand. In Section 3.1 the evolution equations that describe the diffusion of particles in a two-dimensional rectangular lattice is studied in detail. In Section 3.2 the procedures needed to construct a one-dimensional lattice that is topologically equivalent to the two-dimensional rectangular one is described step by step in order to show all possible subtleties appearing in the sequel. In Section 3.3, it is shown how to get rid of border anomalies by wrapping the lattices on a rectangular torus and on a skewed torus, according to whether the lattice is two- or one-dimensional, respectively. Finally, in Section 3.3, conclusions, some generalizations, and perspectives are given.

2. The non-Markovian discrete stochastic evolution updating: basic definitions

In Ref. [7] the basic definitions for a non-Markovian discrete stochastic evolution updating were given and reproduced below, with the appropriate modifications according to the present context, for the sake of completeness and in order to provide the basic framework used in the next sections. A d-dimensional lattice Λ, which can be simplified using some periodic boundary conditions, will be considered and a set of complex dynamical variables $\{q_s^{(r)}(\overrightarrow{x}, t)\}$ will be used for describing the value of each dynamical variable in a realization r, in a state or type of dynamical variable s, at spatial coordinate $\overrightarrow{x} = x_{i_1}, \ldots, x_{i_d}$ and at time $t = t_{i_0}$. For the sake of simplicity it was used the notation $q_s^{(r)}(\overrightarrow{x}, t)$ instead of $q^{(r)}_{s,x_{i_1},\ldots,x_{i_d},t_{i_0}}$ because, as explained below in this section, after an average over realizations the set of discrete evolution equations will be replaced by a smooth interpolating function in order to construct partial differential equations after an expansion in a Taylor series. Note that indices like r and s are used due to the necessity of describing evolution equations that are stochastic and of different types. The spatial coordinates and time correspond to the position of the generic site $\overrightarrow{x}$ and to the update that takes place at time t, respectively. s designate the generic value of the set $\{1, \ldots, S\}$, where S is the number of elements of the set. The spacing between sites or lattice constants is $a_1, \ldots, a_d$ and the time between two successive updates is a_0. In order to save printing and without loss of generality, both or one of the two constants will be set equal to one when it is suitable. The number of lattice sites is $N = N_1N_2 \ldots N_d$ for a hypercubic lattice with a number of sites $N_1, \ldots, N_d$ along each dimension (see Fig. 1 for the case $d = 2$) and the length of the lattice corresponding to each side is $L_k = a_kN_k$ for $k = 1, \ldots, d$. The evolution equation for the set of dynamical variable can be expressed, in the following general form

$$q_s^{(r)}(\overrightarrow{x}, t + a_0) = q_s^{(r)}(\overrightarrow{x}, t) + G_s^{(r)}(X_{l_{01}}, \ldots, X_{l_{0k}}, X_j, X_\xi, t, \ldots, t - l_ka_0), \quad \forall\, s \in \{1, \ldots, S\},\ t \geq 0, \overrightarrow{x} \in \Lambda, \tag{1}$$

where G denotes the set of updating rules that define a given model and $X_{l_{01}}, \ldots, X_{l_{0k}}$ denote the set of complex dynamical variables $\{q_s^{(r)}(\overrightarrow{x}, t)\}$, .., $\{q_s^{(r)}(\overrightarrow{x}, t - l_{0k}a_0)\}$, respectively. The sets of both discrete and continuous stochastic variables that confer stochasticity to the evolution equations are $X_j = \{j\}$ and $X_\xi = \{\xi\}$, respectively. The sets of stochastic variables depend on the particular realization r and previous time $t, \ldots, t - l_{0k}a_0$. The number of previous time is $k + 1$ and the set is $\{l_{0\alpha}\} = \{0, \ldots, k\}$, for any $0 \geq \alpha \geq k$. The stochastic variables are chosen in such a way that all of them are statistically independent and a factorization of each product that contain stochastic variables is then possible. Let us consider a stochastic evolution equation of the form,

$$q_s^{(r)}(\overrightarrow{x}, t + a_0) = q_s^{(r)}(\overrightarrow{x}, t) + \sum_{\{s,k\}} w_{s,k}^{(r)}\, q_s^{(r)}(\overrightarrow{x_k}, t) + \sum_{\{s,s',k,k'\}} w_{\{s,s',k,k'\}}^{(r)}\, q_s^{(r)}(\overrightarrow{x_k}, t)$$
$$\times\, q_{s'}^{(r)}(\overrightarrow{x_{k'}}, t) + \cdots + B_s^{(r)}(\overrightarrow{x_k}, t), \quad \forall s, s' \in \{1, \ldots, S\},\ t \geq 0, \overrightarrow{x_k}, \overrightarrow{x_{k'}} \in \Lambda \tag{2}$$

which correspond to the form that will be considered within the context of the lattice gas model. In order to *derive* deterministic evolution equations, an average over realization of the corresponding stochastic equations of non-Markovian or Markovian type must be done. The number of equations is $M = SN$. The stochastic weights and the dynamical variables, in Eq. (2), are labeled with an index r emphasizing that the value depends on a specific realization. The stochastic weights $w_{s,k}^{(r)}$ and $w_{\{s,s',k,k'\}}^{(r)}$ can, in general, be a complex number but in the examples given below only real weights will be considered. Note also that $B_s^{(r)}(\overrightarrow{x_k}, t)$ was added in order to take into account some corrections corresponding to the borders when periodic boundary conditions are not used.

Beside the evolution equations for the dynamical variables it is possible to construct other evolution equations that are functions of the above mentioned dynamical variables, among others, the evolution equations for the correlations consisting in products of dynamical variables [7]. In the present paper the attention will be focused only on the evolution equation of