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# Functional integral approach to the kinetic theory of inhomogeneous systems

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## HIGHLIGHTS

- A new derivation of the Landau equation for inhomogeneous systems is presented.
- It relies on a functional integral rewriting of the BBGKY hierarchy.
- It appears as a promising alternative to previous methods.

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## ABSTRACT

We present a derivation of the kinetic equation describing the secular evolution of spatially inhomogeneous systems with long-range interactions, the so-called inhomogeneous Landau equation, by relying on a functional integral formalism. We start from the BBGKY hierarchy derived from the Liouville equation. At the order $1/N$, where $N$ is the number of particles, the evolution of the system is characterised by its 1-body distribution function and its 2-body correlation function. Introducing associated auxiliary fields, the evolution of these quantities may be rewritten as a traditional functional integral. By functionally integrating over the 2-body autocorrelation, one obtains a new constraint connecting the 1-body DF and the auxiliary fields. When inverted, this constraint allows us to obtain the closed non-linear kinetic equation satisfied by the 1-body distribution function. This derivation provides an alternative to previous methods, either based on the direct resolution of the truncated BBGKY hierarchy or on the Klimontovich equation. It may turn out to be fruitful to derive more accurate kinetic equations, e.g., accounting for collective effects, or higher order correlation terms.



## 1. Introduction

Recently, the dynamics and thermodynamics of systems with long-range interactions have been a subject of active research [1,2]. The equilibrium properties of these systems, and their specificities such as negative specific heats, various kinds of phase transitions and ensemble inequivalence, are now relatively well understood. However, their dynamical evolution is more complex and many aspects of it need to be improved and exploited in order to obtain explicit predictions. A short historic of the early development of kinetic theory for plasmas, stellar systems, and other systems with long-range interactions is presented in Refs. [3–5]. The main lines of this historic are recalled below, with some complements, in order

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to place our work in a general context. We show in particular how the necessity to develop a kinetic theory for spatially inhomogeneous systems such as those considered in the present paper progressively emerged.

The first kinetic theory describing the statistical evolution of a large number of particles was developed by Boltzmann for a dilute neutral gas [6]. In that case, the particles do not interact except during strong local collisions. The gas is spatially homogeneous and the Boltzmann kinetic equation describes the evolution of the velocity distribution function $f(\boldsymbol{v}, t)$ of the particles under the effect of strong collisions. It can be shown to satisfy a $H$-Theorem corresponding to an increase of Boltzmann's entropy.

Boltzmann's kinetic theory was extended to charged gases (plasmas) by Landau [7]. In that case, the particles interact via long-range Coulombian forces but, because of electroneutrality and Debye shielding [8,9], the interaction is screened on a lengthscale of the order of the Debye length, so that the collisions are essentially local. A neutral plasma is spatially homogeneous and the kinetic equation again describes the evolution of the velocity distribution function $f(\boldsymbol{v}, t)$ of the charges under the effect of close encounters (electrostatic deflections). Since these encounters are weak, one can expand the Boltzmann equation in the limit of small deflections and make a linear trajectory approximation. This leads to the so-called Landau equation [7] which is valid in such a weak coupling approximation. The Landau equation exhibits a logarithmic divergence at small scales due to the neglect of strong collisions (that are rare but that cannot be totally neglected) and a logarithmic divergence at large scales due to the neglect of collective effects, i.e., the dressing of particles by their polarisation cloud (because two like sign charges repel each other and two opposite charges attract each other, a particle of a given charge has the tendency to be surrounded by a cloud of particles of opposite charge). Landau regularised these divergences by introducing rather arbitrarily a lower cut-off at the impact parameter producing a deflection at 90° (Landau length) and an upper cut-off at the Debye length. Collective effects were rigorously taken into account later by Balescu [10] and Lenard [11], leading to the Balescu–Lenard equation. They showed that this equation is valid at the order $1/\Lambda$, where $\Lambda$ is the plasma parameter (number of charges in the Debye sphere). The Balescu–Lenard equation is similar to the Landau equation except that it includes the square of the dielectric function in the denominator of the potential of interaction (in Fourier space). The dielectric function first appeared as a probe of the dynamical stability of plasmas based on the linearised Vlasov equation [12,13]. In the Balescu–Lenard equation the dielectric function accounts for Debye shielding and removes the logarithmic divergence at large scales present in the Landau equation. This amounts to replacing the bare potential of interaction by a dressed potential of interaction. The Landau equation is recovered from the Balescu–Lenard equation by replacing the dielectric function by unity, i.e., by neglecting collective effects. In addition to including the dielectric function, the form of the kinetic equation given by Balescu and Lenard exhibits a local condition of resonance, encapsulated in a Dirac $\delta_{\mathrm{D}}$-function. Resonant contributions are the drivers of diffusion on secular timescales (collisional evolution), as they do not average out. When integrating over this condition of resonance, we recover the original form of the kinetic equation given by Landau.

Self-gravitating systems are spatially inhomogeneous but the early kinetic theories pioneered by Jeans [14] and Chandrasekhar [15–17] were based on the assumption that the collisions (close encounters) between stars can be treated with a local approximation as if the system were infinite and homogeneous. Since a star experiences a large number of weak deflections, Chandrasekhar [18] developed an analogy with Brownian motion. He started from the Fokker–Planck equation and computed the diffusion and friction coefficients in a binary collision theory. This leads to a kinetic equation (usually called the Fokker–Planck equation by astrophysicists) that is equivalent to the Landau equation.[1] The gravitational Landau equation exhibits a logarithmic divergence at small scales due to the neglect of strong collisions and a logarithmic divergence at large scales due to the local approximation or to the assumption that the system is infinite and homogeneous. Strong collisions are taken into account in the treatment of Chandrasekhar which shows, without having to introduce a cut-off, that the small-scale divergence is regularised at the gravitational Landau length. The large-scale divergence is usually regularised by introducing a cut-off at the Jeans length which is the gravitational analogue of the Debye length. The gravitational Landau equation is often thought to be sufficient to describe the collisional dynamics of spherical stellar systems such as globular clusters. However, the treatment based on the local approximation, or on the assumption that the system is infinite and homogeneous, is not fully satisfactory since it leads to a logarithmic divergence. Furthermore, it prevents one from taking into account collective effects, i.e., the dressing of stars by polarisation clouds (because of the gravitational attraction, a star has the tendency to be surrounded by a cloud of stars which increases its effective gravitational mass and reduces its collisional relaxation time). Indeed, if we naively take into account collective effects by introducing the gravitational "dielectric function" in the homogeneous Balescu–Lenard equation (with the sign $-Gm^2$ instead of $+e^2$) we get a strong,

[1] The Landau equation only involves the square of the potential of interaction, so that it keeps the same form for Coulombian and gravitational interactions, except for a change in the prefactor: $(-e^2)^2$ has to be replaced by $(Gm^2)^2$. The kinetic equation derived by Chandrasekhar (see also Ref. [19]), albeit physically equivalent to the Landau equation, did not appear under the same mathematical form because he started from the Fokker–Planck equation $\partial_t f = \partial_{v_i}\partial_{v_j}(D_{ij}f) + \partial_{v_i}(F_i^{\mathrm{fric}} f)$ in which the diffusion tensor is placed after the two velocity derivatives, while the Landau equation can be viewed as a Fokker–Planck equation $\partial_t f = \partial_{v_i}(D_{ij}\partial_{v_j} f) + \partial_{v_i}(F_i^{\mathrm{pol}} f)$ where the diffusion tensor is placed between the two velocity derivatives. From this second rewriting, Landau obtained a symmetric expression of the collision operator from which one can directly deduce all the conservation laws of the system and derive the $H$-theorem for the Boltzmann entropy. Furthermore, Landau derived simultaneously the diffusion and friction coefficients, while Chandrasekhar obtained them from two different calculations and showed a posteriori that they were connected at equilibrium by the Einstein relation. Let us emphasise, however, that the friction force $\boldsymbol{F}^{\mathrm{fric}}$ computed by Chandrasekhar is the true friction force while the friction force $\boldsymbol{F}^{\mathrm{pol}}$ appearing in the Landau equation is the friction due to the polarisation [5].