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Analytic solutions for optimal statistical arbitrage trading

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ABSTRACT

In this paper we derive analytic formulae for statistical arbitrage trading where the security price follows an Ornstein–Uhlenbeck process. By framing the problem in terms of the first-passage time of the process, we derive expressions for the mean and variance of the trade length and the return. We examine the problem of choosing an optimal strategy under two different objective functions: the expected return, and the Sharpe ratio. An exact analytic solution is obtained for the case of maximising the expected return.

1. Introduction

Statistical arbitrage trading has previously been examined by various authors [1–6]. The goal of this type of trading is to develop highly automated trading strategies that take a probabilistic approach to trading. These strategies engage in high frequency trading using algorithms based on stochastic methods to identify price inefficiencies in the market. The use of such an approach has increased substantially in recent years as a greater understanding of the stochastic behavior of financial markets has developed through empirical investigation and phenomenological modeling. This has largely been driven by an increase in interdisciplinary research by physicists and has allowed for the development of increasingly sophisticated models for price behavior.

A common approach when performing this type of trading is to construct a stationary, mean reverting synthetic asset as a linear combination of securities. One example is the method of pairs trading which has been the focus of several recent studies [2,3,7]. This approach allows for the construction of a trading strategy where trades are entered when the process reaches an extreme value and exited when the process reverts to some equilibrium value.

Most commonly used methods for investing do not address the significance of the role of time, mainly due to the fact that modern portfolio theory [8,9] is based on single-period models. However, when considering statistical based strategies that engage in high frequency trading, the time between trades, i.e. trading frequency, becomes an important quantity to consider. The importance of the role that time plays in financial markets has been explored by many studies, for instance: data seasonality [10,11], market activity time [12–14], and waiting times between orders and trades [15–19]. In the context of trading strategies it is crucial to consider not only the return per trade but also the time over which the returns take place. In such a setting it is imperative to consider transaction costs, because whether inefficiencies can be successfully traded depends on the cost of trading. Continuous time trading strategies were presented in Ref. [5] to provide a mathematical framework for the construction and analysis of statistical arbitrage methods. It was shown that in this framework that optimal strategies balance return per trade and transaction cost with the stochastic trading frequency.

When implementing these types of strategies, speed of computation is vital, as calculations are often required to be performed in real time. A high frequency trading desk may perform thousands of transactions each day on hundreds of different securities. In this situation, numerical methods, such as simulation or quadrature, may not be fast enough to update

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calculations within the required time constraints. In such an environment there is a need for analytic and approximate solutions.

In this paper we present analytic formulae and solutions for calculating optimal statistical arbitrage strategies with transaction costs. We assume that the traded security is described by an Ornstein–Uhlenbeck process [20]. The resulting analysis provides a mathematical model which can be used to explore the relationships between variables and offer insight into the dynamics of trading strategies. We construct a continuous trading strategy for the Ornstein–Uhlenbeck process and express the trade length and return of the strategy in terms of the first-passage time of the process. Using known solutions for the first-passage time, we derive analytic solutions for: the expected return; the variance of the return; and the expected trade length of the strategy. Optimal trading strategies can be found by constructing objective functions that are expressed in terms of the expected value and variance of the return. We derive an analytic solution for the strategy that maximises expected return. The solution is shown to satisfy a real valued integral equation. We present an approximate solution via a Taylor series expansion that is in close agreement with the exact solution. We prove that for optimal trading, the trading bands are symmetric about the mean of the traded security. We formulate expressions for the Sharpe ratio [9] of the strategy and show how it can be maximised. Results are illustrated using an example from an earlier work, in which, numerical methods were used to evaluate first-passage time distributions. The model allows for the examination of the impact of transaction cost on trade length and return, in order to determine whether a strategy can be successful.

The rest of the paper is as follows. In Section 2 we define the continuous time trading strategy for the Ornstein–Uhlenbeck process. The trade length, expected return, and variance of the return are formulated in terms of the first-passage time of the Ornstein–Uhlenbeck process. We use known expressions for the moments of the first-passage time to derive analytic formulae for the mean and variance of the trade length and return. In Section 3 we construct optimal trading strategies for the Ornstein–Uhlenbeck process by maximising the expected return and maximising the Sharpe ratio. An analytic solution to the problem is obtained in the case of maximising the expected return. In Section 4 we present the results for the optimal strategies applied to real world data. Section 5 concludes and summarises the main results of the paper.

2. Continuous time trading

A continuous trading strategy comprises a sequence of individual trades performed on a continuous time stochastic process. Consequently, many of the important quantities related to the trading strategy are functions of the frequency at which these trades take place. The trading frequency is specified by how many times the strategy trades per unit of time. This value is dependent on how long it takes in total to move from one trade entry point to the next, passing though the exit point along the way. We model the price of the traded security p_t as,

$$p_t = \mathrm{e}^{X_t}; \qquad X_{t_0} = x_0, \tag{1}$$

where X_t satisfies the following stochastic differential equation,

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \eta \mathrm{d}W_t, \tag{2}$$

where $\alpha > 0$, $\eta > 0$, and W_t is the Wiener process. A continuous time trading strategy is defined by entering a trade when $X_t = a$, exiting the trade at $X_t = m$, and waiting until the process returns to $X_t = a$, to complete the trading cycle. Such a strategy can be thought of as periodic, since the actions are repeated between trade entry points. However, since X_t is a stochastic process, the time taken to complete the trade cycle will be a random variable $\mathcal{T}$. We refer to $\mathcal{T}$ as the total trade length. Thus, the behavior of $\mathcal{T}$ will largely determine the properties of the strategy.

We assume that $a < m$ and decompose the total trade length into sub-intervals,

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2,$$

where $\mathcal{T}_1$ is the time taken for the process to travel from a to m and $\mathcal{T}_2$ is the time taken from m back to a. Here the variables $\mathcal{T}_1, \mathcal{T}_2$ can be identified as first-passage times for the process X_t. Since X_t is a Markov process, the passage times $\mathcal{T}_1, \mathcal{T}_2$ are independent. Furthermore, it is well known that the first-passage time for the Ornstein–Uhlenbeck process has finite mean and variance [21]. Thus the mean and variance of $\mathcal{T}$ may be written as,

$$\mathbb{E}[\mathcal{T}] = \mathbb{E}[\mathcal{T}_1] + \mathbb{E}[\mathcal{T}_2], \tag{3}$$

$$\mathbb{V}[\mathcal{T}] = \mathbb{V}[\mathcal{T}_1] + \mathbb{V}[\mathcal{T}_2]. \tag{4}$$

Expressions for the return per unit time and variance of the return per unit time of the strategy, can be formulated in terms of the trading frequency. Let $r(a, m, c)$ be the return per trade as a function of entry and exit levels and transaction cost. Then, the expected value and the variance of the return per unit time for the strategy are given by,

$$\mu(a, m, c, t) = r(a, m, c)\mathbb{E}\left[N_t\right]/t, \tag{5}$$

$$\sigma^2(a, m, c, t) = r(a, m, c)^2\mathbb{V}\left[N_t\right]/t, \tag{6}$$

where the variable N_t represents the number of trades over a time interval of length t. In this representation, N_t is the counting process for a renewal process where the interarrival times are given by the trade cycle length $\mathcal{T}$. Since the value