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Stochastic response and bifurcation of periodically driven nonlinear oscillators by the generalized cell mapping method

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HIGHLIGHTS

- Extended short-time Gaussian approximation to periodically driven systems.
- Introduced construction of the transition probability matrix over one period.
- Studied stochastic response by the generalized cell mapping method.
- Studied transient and steady-state responses of SD oscillator.
- Found stochastic P-bifurcation of SD oscillator.

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ABSTRACT

The stochastic response of nonlinear oscillators under periodic and Gaussian white noise excitations is studied with the generalized cell mapping based on short-time Gaussian approximation (GCM/STGA) method. The solutions of the transition probability density functions over a small fraction of the period are constructed by the STGA scheme in order to construct the GCM over one complete period. Both the transient and steady-state probability density functions (PDFs) of a smooth and discontinuous (SD) oscillator are computed to illustrate the application of the method. The accuracy of the results is verified by direct Monte Carlo simulations. The transient responses show the evolution of the PDFs from being Gaussian to non-Gaussian. The effect of a chaotic saddle on the stochastic response is also studied. The stochastic P-bifurcation in terms of the steady-state PDFs occurs with the decrease of the smoothness parameter, which corresponds to the deterministic pitchfork bifurcation.

1. Introduction

The stochastic response analysis has been a classic and significant research topic in nonlinear dynamical systems [1–3]. In particular, when a nonlinear oscillator is driven by periodic and random excitations, it exhibits complex stochastic response phenomena including multiple steady-state responses [4], chaotic responses [5] and stochastic bifurcations such as P-bifurcation [6]. Stochastic P-bifurcation is defined by the sudden change of the shape of the steady-state probability density function (PDF) [7]. As an important characterization of the stochastic response, the PDF always attracts the attention of many researchers. The PDF of a Markov response process is governed by the Fokker–Planck–Kolmogorov (FPK) equation [8,9].

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However, the exact solutions of FPK equations are available only in a few special classes of lower order nonlinear stochastic systems. For the nonlinear systems driven by periodic and random excitations, the FPK equation is even more difficult to deal with. Approximate and numerical procedures are usually resorted to solve it. This paper presents the generalized cell mapping (GCM) method based on the short-time Gaussian approximation (STGA) scheme to attack this problem.

The GCM method was first developed by Hsu [10] in 1981. It allows each cell in the discrete cell state space to have multiple image cells. The relationship between a cell and its image cells is described by the one-step transition probabilities, leading to a Markov chain. During the past decades, the GCM method has been widely used for global analysis and bifurcation of strongly nonlinear systems [11–14]. The GCM method is also applied to study the response PDFs [15–18] and the first-passage time probability [19,20] of stochastic dynamical systems. Sun and Hsu [21] proposed a STGA scheme to evaluate the one-step transition probabilities in the GCM method for nonlinear systems under Gaussian white noise excitations. The GCM/STGA method was successfully extended to the systems with discontinuous nonlinearities in the form of dry friction damping [22]. With the help of STGA, the computational time for the GCM method can be greatly reduced, while the accuracy of the solution is maintained.

The GCM/STGA method can be viewed as an extension of the path integral solution method [23,24]. The path integral solution method aims to evolve the PDF from an initial distribution by repeatedly using a short-time transition probability density function (TPDF), which is based upon a Taylor expansion of the solution to the first order approximation. While the short-time TPDF in the GCM/STGA method is constructed with the mean and covariance obtained by integrating the moment equations with Gaussian closure.

In the present paper, the GCM/STGA method is illustrated with the periodically driven smooth and discontinuous (SD) oscillator originally proposed by Cao and coworkers [25]. This is an archetypal system to investigate the transitions from smooth to discontinuous dynamics, depending on the value of the smoothness parameter. It can be used to study a snap-through truss system composed of a mass linked by a pair of inclined elastic spring being capable of resisting both tension and compression [26]. The deterministic SD oscillator has been studied extensively in recent years, including the response of discontinuous case [27], codimension-two bifurcation [26], piecewise linear approximation [28], resonant behaviors in discontinuous case [29], and interior crisis [30]. Since the SD oscillator is inevitably affected by various random disturbances, Yue et al. [31] studied the global analysis of stochastic bifurcation in a SD oscillator under bounded noise excitations. The stochastic analysis of the SD oscillator under periodic and random excitations has been seldom reported in the literature.

The rest of this paper is organized as follows. The GCM/STGA method for periodically driven stochastic systems is presented in Section 2. In Section 3, the stochastic differential equation of the SD oscillator is studied. The moment equations with Gaussian closure are derived. Then, Section 4 focuses on the stochastic response analysis of the SD oscillator. The results of both transient and steady-state PDFs are presented. P-bifurcations are illustrated by the variation of the PDFs with the smoothness parameter. Finally, the paper ends with some conclusions in Section 5.

2. Response of periodically driven stochastic systems

Consider a multi-dimensional nonlinear dynamical system subject to periodic and Gaussian white noise excitations. The system is governed by the Itô stochastic differential equation

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}, t)\mathrm{d}t + \boldsymbol{\sigma}(\mathbf{X})\mathrm{d}\mathbf{B}(t), \tag{1}$$

where $\mathbf{X}(t) = [X_1(t), X_2(t), \ldots, X_N(t)]^T \in \mathbf{R}^N$ is an N-dimensional stochastic process representing the response of the system, and it is known to be Markovian. The superscript T denotes transpose. $\mathbf{f}(\mathbf{X}, t) : \mathbf{R}^N \times R^1 \mapsto \mathbf{R}^N$ is the $N \times 1$ drift vector function such that $\mathbf{f}(\mathbf{X}, t) = \mathbf{f}(\mathbf{X}, t + T)$, where T is the period. $\boldsymbol{\sigma}(\mathbf{X}) : \mathbf{R}^N \mapsto \mathbf{R}^L$ is an $N \times L$ diffusion matrix. $\mathbf{B}(t) = [B_1(t), B_2(t), \ldots, B_L(t)]^T \in \mathbf{R}^L$ is a standard L-dimensional vector Wiener process with the following properties

$$E[\mathrm{d}\mathbf{B}(t)] = \mathbf{0}, \qquad E[\mathrm{d}\mathbf{B}(t)\mathrm{d}\mathbf{B}^T(t')] = \begin{cases} \mathbf{I}\mathrm{d}t, & t = t', \\ \mathbf{0}, & t \neq t', \end{cases} \tag{2}$$

where $\mathbf{I}$ is an $L \times L$ identity matrix. The formal derivative of Wiener process $\mathbf{B}(t)$ is the L-dimensional standard Gaussian white noise.

2.1. The GCM method

Because $\mathbf{f}(\mathbf{X}, t)$ in Eq. (1) is a periodic function in time, the response $\mathbf{X}(t)$ can be described by a point mapping over the period T given by $\mathbf{G} : \mathbf{X}(t) \mapsto \mathbf{X}(t + T)$. This is known as the Poincaré map. Assume that $\mathbf{X}(t_0) = \mathbf{x}_0$ with probability one. The PDF of $\mathbf{X}(t)$ at time $t = t_0 + T$ conditional on the initial condition is denoted as $q(\mathbf{x}, t_0 + T|\mathbf{x}_0, t_0)$. This conditional probability density function describes the statistics of the map $\mathbf{G}$. From the FPK equation, we can show that $q(\mathbf{x}, t + T|\mathbf{x}_0, t_0 + T) = q(\mathbf{x}, t|\mathbf{x}_0, t_0)$ is also a solution. By choosing $t_0 = nT (n = 0, 1, 2, \ldots)$ and $t = (n + 1)T$, we have $q(\mathbf{x}, (n+1)T|\mathbf{x}_0, nT) = q(\mathbf{x}, T|\mathbf{x}_0, 0)$. Suppose that $p(\mathbf{x}, nT)$ denotes the response PDF of the process $\mathbf{X}(t)$ at time nT. By the definition of the conditional probability, we have

$$p(\mathbf{x}, (n + 1)T) = \int_{\mathbf{R}^N} q(\mathbf{x}, T|\mathbf{x}_0, 0)p(\mathbf{x}_0, nT)\mathrm{d}\mathbf{x}_0, \quad n = 0, 1, 2, \ldots \tag{3}$$