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Quantum decomposition of random walk on Cayley graph of finite group[☆]

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HIGHLIGHTS

- I first introduce the decomposition of quantum random walk operator by the help of quantum probability approach.
- We then propose the state decomposition of quantum random walk with the help of eigenvalue and eigenvector.
- As an application we then discuss the probability distribution for the quantum random walk on Cayley graph.

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ABSTRACT

In the paper, A quantum decomposition (QD, for short) of random walk on Cayley graph of finite group is introduced, which contains two cases. One is QD of quantum random walk operator (QRWO, for short), another is QD of Quantum random walk state (QRWS, for short). Using these findings, I finally obtain some applications for quantum random walk (QRW, for short), which are of interest in the study of QRW, highlighting the role played by QRWO and QRWS.

1. Introduction

Throughout the paper I will restrict our study to discrete time QRW on Cayley graphs of finite group. The theory of QRW on graphs is fundamental to mathematics, physics, and computer science [1,2], as it provides a beautiful mathematical framework to study stochastic process and its applications. An appealing well-studied classical idea in statistics and computer science is the method of QRW [3,4] or [5], which is the quantum analogue of classical random walks, has been studied in a flurry of works. Some of these works studied the problem in the important context of algorithmic problems on graphs and suggested that quantum walks is a promising algorithmic technique for designing future quantum algorithms.

With the quantum coin reflecting the graph structure, QRW may also be seen as a nice tool for classifying, or at least describing the structure of Cayley graphs, groups and related objects. Several important classes of graphs studied in QRW include the group-theoretic Cayley graphs, such as Cayley graph of Abelian group, Cayley graph of free group, Cayley graph of cyclic group, and so on.

The main purpose of this paper is to show QD of QRWO and QRWS on Cayley graph of finite group. The QD of QRWO includes QD of shift operator $\hat{S}$ and coin operator C, while the QD of QRWS is a type of state superpositions.

There are a lot of works about QRW on Cayley graphs of finite group (see Refs. [6–9]). And Cayley graphs capture strong group-theoretic ingredients of important problems, such as Graph Isomorphism. Since most of these graphs are regular,

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the classical random walks (CRW, for short) on them are known to converge or to mix towards the uniform stationary distribution. The mixing properties of continuous-time QRW on the same graphs were found to exhibit non-classical behaviour [5,10–12].

Consider, for example, a free group over three generators, a, b, and c. The elements are uniquely described by words in the alphabet $\mathcal{A} = \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$. Define a map

$$\mathcal{A}^* \to \mathcal{A}^*$$

which substitutes b for c and vice versa for words beginning with an a and leaves all other words intact. Such a map induces a graph automorphism of the Cayley graph but is not a group automorphism itself.

On the contrary, an automorphism of a graph is a permutation of its vertices such that it leaves the graph unchanged. The set of all such permutations is the automorphism group of the graph.

Taking advantage of these results we get some associated implemented results of Cayley graph and applications of QRW on the Cayley graph over finite group.

The paper is organized as follows: In Section 2 we give a brief review of the main characteristic of Cayley graphs. In Section 3 we introduce some basic notions and notations for QRW, see, e.g., Refs. [1–3,13–20], subsequently, we focus on studying QD of QRWO and QRWS on Cayley graph of finite group. In Section 4 as two applications to limit distribution of QRW and time averaged proability distribution of QRW.

2. Cayley graphs and its adjacency matrix

In this section we give a brief outline of some of the main features of Cayley graphs, such as adjacency matrix, stratification and orthonormal basis of strata. In general, Cayley graphs are defined as follows:

Definition 2.1. Let G be a discrete group finitely generated by a set S. The Cayley graph $\Gamma = \Gamma(G, S)$ is a directed graph (G, E), where the set of vertices is identified with the set of elements of G and the set of edges is

$$E = \{(g, s)|g \in G, s \in S, gs^{-1} \in S\}.$$

Lemma 2.2 ([19]). *A Cayley graph (G, E) is regular with degree $2N$.*

Example 2.3. The added group $\mathbb{G}^N$ furnished with the standard generators

$$g_{\pm 1} = (\pm 1, 0, \ldots, 0), \ldots, g_{\pm N} = (0, 0, \ldots, \pm 1),$$

is the N-dimensional integer lattice with the degree $2N$.

Example 2.4. Let G_N be the free group with N free generators $g_1, \ldots, g_N$. We write $g_{-i} = g_i^{-1}$ for simplicity. The Cayley graph $(G_N, \{g_{\pm 1}, \ldots, g_{\pm N}\})$ is a homogeneous tree with degree $2N$. In particular, a homogeneous tree with degree two is one-dimensional integer lattice.

A Cayley graph is a pair $\Gamma = (V, E)$ where V is a non-empty set and E is a subset of $\{(i, j); i, j \in V, i \neq j\}$. Elements of V and of E are called vertices and edges, respectively. Two vertices $i; j \in V$ are called adjacent if $(i, j) \in E$, and in that case we write $i \sim j$. For a Cayley graph $\Gamma = (V, E)$ we define the adjacency matrix $A = (A_{ij})_{i,j \in V}$ by

$$A_{i,j} := \begin{cases} 1, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Obviously, (i) A is symmetric; (ii) an element of A takes a value in $\{0, 1\}$; (iii) a diagonal element of A vanishes. Conversely, for a non-empty set V, a graph structure is uniquely determined by such a matrix indexed by V.

The degree or valency of a vertex $i \in V$ is defined by

$$k(i) = |\{j \in V; i \sim j\}|, \tag{2.2}$$

where $|\cdot|$ denotes the cardinality. A finite sequence $i_0; i_1; \ldots; i_n \in V$ is called a walk of length n (or of n steps) if $i_{k-1} \sim i_k$ for all $k = 1, 2, \ldots, n$. In a walk some vertices may occur repeatedly. Unless otherwise stated, we always assume that a graph under discussion satisfies:

(a) (connectedness) any pair of distinct vertices is connected by a walk;
(b) (local boundedness) $k(i) < \infty$ for all $i \in V$; In fact, the examples in this paper satisfy the following condition which is stronger than (b):
(b′) (uniform boundedness) $\sup_{i \in V} k(i) < \infty$.

Suppose that $l^2(V)$ denotes the Hilbert space of $\mathbb{C}$-valued square-summable functions on V, and $\{|i\rangle; i \in V\}$ becomes a complete orthonormal basis of $l^2(V)$. The adjacency matrix is considered as an operator acting in $l^2(V)$ in such a way that

$$A|i\rangle = \sum_{i \sim j} |j\rangle, \quad i \in V. \tag{2.3}$$

Then, A becomes a self-adjoint operator equipped with a natural domain. As is easily checked, "(b′)" is a necessary and sufficient condition for A to be a bounded operator on $l^2(V)$.