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On a nonstandard Brownian motion and its maximal function

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HIGHLIGHTS

- A nonstandard construction of Brownian motion (the Wiener walk) is given.
- Known results for the random walk are given a nonstandard version.
- The results are extended to the Wiener walk.
- The work is entirely based on Nelson's Radically Elementary Probability Theory.

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ABSTRACT

This article uses Radically Elementary Probability Theory (REPT) to prove results about the Wiener walk (the radically elementary Brownian motion) without the technical apparatus required by stochastic integration. The techniques used replace measure-theoretic tools by discrete probability and the rigorous use of infinitesimals. Specifically, REPT is applied to the results in Palacios (*The American Statistician*, 2008) to calculate certain expectations related to the Wiener walk and its maximal function. Because Palacios uses mostly combinatorics and no measure theory his results carry over through REPT with minimal changes. The paper also presents a construction of the Wiener walk which is intended to mimic the construction of Brownian motion from "continuous" white noise. A brief review of the nonstandard model on which REPT is based is given in the Appendix in order to minimize the need for previous exposure to the subject.

1. Introduction

Palacios [1] showed how to use elementary results from Feller [2] in order to compute certain expectations associated with the simple symmetric random walk. He mentions that his results could be extended to Brownian motion but the necessary limiting operations would require advanced material outside the scope of Feller. The advanced, measure-theoretic, material is required in the transition from discrete-time to continuous-time processes in order to do rigorous stochastic limiting operations involving the derivative of Brownian motion.

Could we overcome these technicalities about the derivative of Brownian motion by means of elementary mathematics? The answer depends on what we mean by "elementary". What Palacios means by "elementary" is material from Feller consisting of combinatorics and Stirling's approximation which is not enough to deal with Brownian motion. However, a *nonstandard model* proposed by Edward Nelson [3] and known as Radically Elementary Probability Theory (REPT) provides a way for studying probability at a general level without requiring measure theory. REPT covers basic probability and stochastic processes up to the martingale central limit theorem using a model that is entirely based on finite sample spaces. Recent advances in this field include the study of interacting particle systems [4] and applications of stochastic calculus to both mathematical physics and financial engineering [5].

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Here the term "model" is to be understood in the sense of model theory [6], and also in the Physics sense, a structure representing something not directly observable, and not in the sense of a mathematical model, say a set of equations, describing an observable physical system. In either model (REPT or conventional probability) we play with idealizations:

> "Nonstandard analysis is a powerful addition to classical mathematics. Statements and proofs in it can be reduced to classical statements and proofs, and often the nonstandard proofs are much shorter and easier to understand. *But the new concepts are in general new, not reducible to classical mathematics.* (…) Mathematics is our invention, and we can have infinitesimals or not, as we choose. The only constraint is consistency". (Nelson [7]; emphasis added.)

The question is which abstraction (unlimitedness *vs.* infinity) we wish to use. The use of infinite sequences requires real analysis whereas the use of infinitesimals and their reciprocals (unlimited numbers) requires nonstandard mathematics. Neither model is trivial but the simplification provided by REPT is of particular use for those interested in the essential aspects of probability theory without long technical detours.

The Appendix below summarizes the basic notions of Nelson's nonstandard model but the reader can safely proceed without it and trust his intuition when encountering concepts which are not part of conventional analysis (infinitesimality, near equality, unlimitedness, etc.). After a few definitions regarding stochastic processes and the Wiener walk (Section 2) I will illustrate the reach of REPT by proving three new results.

The first result (Theorem 2.1) is a characterization of the radically elementary analogue of Brownian motion, the *Wiener walk* [8,9,3], in terms of difference equations (i.e. radically discretized differential equations). A more sophisticated characterization is given by radically elementary versions of Girsanov's theorem in Refs. [8,5] but it is not needed here.

The second proposition (Theorem 2.2) is just an adaptation of Palacios' main results stated in terms of nonstandard concepts.

Finally, the third result (Theorem 2.3) picks up where Palacios left off to show how his results can be extended to the Wiener walk without the "advanced notions" he wanted to avoid. Particularly, REPT is combined with the results in Palacios to calculate $\mathrm{E}\,W(t)$, $\mathrm{E}\,M(t)$, $\mathrm{E}\,W^2(t)$, $\mathrm{E}\,M^2(t)$ and the correlation between $W(t)$ and $M(t)$ where $W(t)$ represents the Wiener walk and $M(t)$ is its maximal function at time t.

2. The Wiener walk

2.1. Introduction

In REPT, a *stochastic process* X is a random sequence indexed by a finite time set T and defined over a finite probability space (Ω, pr), $X : T \times \Omega \mapsto \mathbb{R}$. With the usual discarding of the argument $\omega \in \Omega$ we can denote X at time $t \in T$ either by $X(t)$ or by X_t.

Let $T = \{a, \dots, b\}$. For t in $T' = T \setminus \{b\}$ we write $t + \mathrm{d}t$ for its successor and define $\mathrm{d}X(t) = X(t + \mathrm{d}t) - X(t)$.

Remark 2.1. Within REPT, one may think of a "continuous-time process", such as Brownian motion, when $T = \{0, \dots, 1\}$ is a *near interval*, that is with infinitesimal increments ($\mathrm{d}t \simeq 0,\ \forall t$), and a "discrete-time process" when $T = \{1, 2, \dots, \nu\}$ with $\nu \simeq \infty$, but in either case T is a *finite* subset of $\mathbb{R}$. The state space Ω on which each $X(t)$ is defined is also a *finite* subset of $\mathbb{R}$ which can also be nearly continuous (but still discrete) by having infinitesimal spacing.

A process W on the near interval $T = \{a, \dots, b\}$ such that

I. $W(a) = 0$;
II. $\mathrm{d}W(a), \dots, \mathrm{d}W(b - \mathrm{d}t)$ are independent;
III. $\mathrm{d}W(t) = \pm\sqrt{\mathrm{d}t}$ with probability 1/2;

is called the *Wiener walk*. It is the radically elementary equivalent of Brownian motion and it appears in REPT's functional central limit theorem [3, Chapter 18] though Nelson uses martingale theory to motivate it. Note that there is no need to build a process as the limiting case of a random walk. The Wiener walk has already been built with both its step lengths and time intervals infinitesimal.

In conventional probability much of the motivation to study Brownian motion $B(t)$ comes from stochastic differential equations. Important equations in Physics and Finance are formulated as linear first-order stochastic differential equations which take the derivative $\dot{B}(t)$ as input, such as Langevin's velocity process,

$$a\dot{Y}(t) + bY(t) = \dot{B}(t).$$

However the derivative of $B(t)$, does not exist as a regular stochastic process and this is usually overcome by physical interpretation or by analogies with the discrete case. Doob [10] notes that such equations are to be interpreted "symbolically", since $\dot{B}(t)$ does not exist in the usual sense. Technically, of course, there is nothing wrong or pathological: $\mathrm{d}B(t)$ is not a separate mathematical object but always part of a (stochastic) integral. However the mathematical apparatus needed to fully grasp this concept is substantial. Notice that Karlin and Taylor [11, Chapter 7] make a disclaimer after giving some justification for the fact that $\int_0^t [\mathrm{d}B(\tau)]^2 = t$: "A typical feeling on seeing this for the first time is disbelief accompanied by a strong desire to check the analysis carefully and preclude the possibility of error. The differential formula $[\mathrm{d}B(t)]^2 = \mathrm{d}t$