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A Fisher-gradient complexity in systems with spatio-temporal dynamics

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HIGHLIGHTS

- Statistical complexity measures (SCM) for spatio-temporal systems studied.
- We define a benchmark of complexity.
- We focus in particular on the Collective Motion model.
- We analyse LMC's, Autocorrelation, and Fisher-gradient complexities.

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ABSTRACT

We define a benchmark for definitions of complexity in systems with spatio-temporal dynamics and employ it in the study of Collective Motion. We show that LMC's complexity displays interesting properties in such systems, while a statistical complexity model (SCM) based on autocorrelation reasonably meets our perception of complexity. However this SCM is not as general as desirable, as it does not merely depend on the system's Probability Distribution Function. Inspired by the notion of Fisher information, we develop a SCM candidate, which we call the Fisher-gradient complexity, which exhibits nice properties from the viewpoint of our benchmark.

1. Introduction

Perfect disorder maximizes missing-information, in the same fashion as entropy does, but it is actually not much more complex than perfect order, which minimizes entropy. As Crutchfield noted in 1994, *Physics does have the tools for detecting and measuring complete order equilibria and fixed point or periodic behaviour and ideal randomness via temperature and thermodynamic entropy or, in dynamical contexts, via the Shannon entropy rate and Kolmogorov complexity. What is still needed, though, is a definition of structure and way to detect and to measure it* [1].

Seth Lloyd counted as many as 40 ways to define complexity, none of them being completely satisfactory. A major breakthrough came from the definition of statistical complexity proposed by López-Ruiz, Mancini and Calbet (LMC) [2]. Although not without problems [3,4], LMC's complexity clearly separated and quantified the contributions of entropy and structure. LMC measured structure via the concept of *disequilibrium*. Building on this proposal, Kowalski et al. [4] refined the definition of disequilibrium.

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One should note that while entropy is a general concept that can be applied across a wide range of model families, this is not the case with measures of structure. For them one needs to know, in advance, what to look for. The LMC proposal can be regarded as a way to provide a very general definition of structure, which does an excellent job for many systems, particularly those involving time series, but also 1D spatial systems [5].

We are, however, specifically interested in a statistical complexity measures (SCM) for models with spatial dimensions. This includes dynamical PDE-based models, such as Navier–Stokes, on the one hand, and Agent-Based Models (ABM), such as Collective Motion [6], on the other. Here we use the adjective *dynamical* because the structures we are interested in are easily recognized (at least visually) by studying velocity fields. Other models of interest are static (i.e., not characterized by a velocity field, but rather from scalar quantities such as density or spin). Examples of these models include PDE-based models such as Reaction–Diffusion, and Cellular Automata (e.g. Ising models).

Within this framework one may try to be more specific in the definition of structure. In a previous paper [7], we showed that, for these systems, a good candidate for appropriately capturing the structural component in the definition of complexity is a correlation (specifically the velocity autocorrelation field in the case of the Collective Motion model).

Density fields as a proxy for structural information were studied in the case of reaction–diffusion models [8] and atoms [9].

The problem with these approaches is that the definition of structure is quite ad hoc, limited to families of models. In this paper we explore the possibility to define a more general statistical complexity measure. We aim for a definition which is more specific to spatial systems than the one based on disequilibrium, but still general in this context, that so it does not depend on the existence of specific fields such as velocity or density.

We start by observing a common feature of perceived complexity in spatial systems characterized by a velocity field: both perfect order and perfect disorder are characterized by vanishing spatial derivatives and time derivatives of the velocity probability distribution function (PDF) at mesoscopic scales. A well grounded information measure involving derivatives is the Fisher information one [10], which has already been used to analyse electronic structure [11,12]. We aim for a measure which is independent from the chosen coordinate system and takes into account all spatial derivatives.

The paper is structured as follows. In Section 2 we define the procedure we employ to define a PDF for the Collective Motion model, and we provide definitions of different candidates to SCM based on the PDF definition. In Section 3 we define a benchmark that will allow us to quantify the assessment of complexity for models with spatial coordinates. We also recall the Collective Motion model, which will serve to test the SCM candidates against our complexity benchmark. Section 4 presents the results of such test, which we discuss in Section 5 to show that LMC's complexity cannot be trivially extended to spatial systems, and that a SCM inspired by Fisher information does a good job at capturing the essence of complexity and is generalizable to any system with spatio-temporal dynamics.

2. Definitions

It is important to acknowledge that the perception of complexity is deeply entangled with the scale of measurement. Therefore, we should aim at measuring complexity at different scales. The idea of studying complexity as a function of scale is not new, as represented for instance in the concepts of *complexity profile* [13,14] and *d-diameter complexity* [15].

A reference microscale, in Agent-Based Models, is given by the typical (average) separation of two agents (δ). Larger scales (mesoscale and macroscale) may be characterized as certain multiples, with values depending on the system, of this basic scale. The calculation of metrics at a certain scale l at point x, y[1] are done by considering all agents falling into a ball of radius $l/2$ around x, y.

We focus on models characterized by a velocity field, which we will illustrate with the 2D Collective Motion model. That is, we consider the state of the system as $s = \{v^i_\alpha\}$, where i indexes spatial coordinates and α individual agents. The same procedure is feasible for other fields (density, spin, and so on). One needs to properly characterize the fields that, in each case, better define the structures representing complexity.

Following [6], and re-scaling so we get a positive value, we define the velocity autocorrelation of a pair of agents α and β as

$$A_{\alpha\beta} \equiv \frac{1}{2} + \frac{v^x_\alpha v^x_\beta + v^y_\beta v^y_\alpha}{v^2_\alpha + v^2_\beta}. \tag{1}$$

For each particle α we compute the average velocity autocorrelation with all neighbours β within radius $l/2$[2] so that we get $A_{(l)}(x, y)$.

This definition of velocity correlation guarantees positivity, ranging from 0 in the anti-parallel case (−1 in the standard definition of velocity correlation) to 1 in the parallel case (also 1 using the standard definition).

[1] Let us develop, without loss of generality, the expressions for 2D Cartesian systems.
[2] In practice this is only reasonable for small scales. For large scales we need to sample the pairs we take into account.