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Characterizing Detrended Fluctuation Analysis of multifractional Brownian motion

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HIGHLIGHTS

- Our work characterizes a single exponent estimator like DFA when applied to mBm.
- DFA estimates a time averaged Hurst exponent in systems: this assertion is verified.
- We identify parameters that can impact the robustness of DFA.
- Results serve as benchmark for using DFA as sliding window Hurst exponent estimator.

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ABSTRACT

The Hurst exponent (H) is widely used to quantify long range dependence in time series data and is estimated using several well known techniques. Recognizing its ability to remove trends the Detrended Fluctuation Analysis (*DFA*) is used extensively to estimate a Hurst exponent in non-stationary data. Multifractional Brownian motion (*mBm*) broadly encompasses a set of models of non-stationary data exhibiting time varying Hurst exponents, $H(t)$ as against a constant H. Recently, there has been a growing interest in time dependence of $H(t)$ and sliding window techniques have been used to estimate a local time average of the exponent. This brought to fore the ability of *DFA* to estimate scaling exponents in systems with time varying $H(t)$, such as *mBm*. This paper characterizes the performance of *DFA* on *mBm* data with linearly varying $H(t)$ and further test the robustness of estimated time average with respect to data and technique related parameters. Our results serve as a bench-mark for using *DFA* as a sliding window estimator to obtain $H(t)$ from time series data.

1. Introduction

Statistical properties such as trends and correlations of complex phenomena are important in the study of non-equilibrium phenomena such as extreme events. Due to the non-equilibrium nature of complex driven systems, general statistical analysis tools cannot be readily applied to them. Long range dependence (*LRD*) in data is a key feature [1] and is studied in data from diverse physical systems such as temperature records, river flows, heart beat variability, and space weather, [2–11].

Rescaled range analysis (*R/S*) [12] and fluctuation analysis (*FA*) [13] are statistical tools developed to estimate the variability of time series through estimation of Hurst exponent, H [14], a statistic which is directly related to the scaling

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in autocorrelation functions, and, also to the fractal dimension of the time series data. While the scaling exponent, H, is equal to 0.5 for uncorrelated white noise, many natural systems demonstrate values close to 0.7 [15].

These techniques, however, fail to estimate H in non-stationary data. More recently, Detrended Fluctuation Analysis (*DFA*) [16], which is widely considered a better technique than either *R/S* or *FA* due to its capability to detrend a time series data while estimating H, making it viable for non-stationary systems. With increased use of *DFA* technique, its limitations in detrending capabilities are evident [17] and there is need for better alternative detrending schemes for data with atypical trends e.g., *trends that are not addressable by polynomial detrending* [18]. In spite of its purported shortcomings, *DFA* is recognized as an efficient Hurst exponent estimation technique because it utilizes detrending to estimate over lesser number of averages than *FA*.

Fractional Brownian motion (*fBm*), a generalization of Brownian motion, is a quintessential theoretical model for the Hurst effect [19]. Since its discovery, there has been an interest in modeling physical systems as *fBm*. However, it was quickly realized that imposing a uniform H over the span of the data is in fact a restricting condition as uniform level of *LRD* in real life data is uncommon. Multifractional Brownian motion (*mBm*) is a generalization of *fBm* relaxing this condition [20], allowing for variable degrees of self-similarity with non-stationary increments i.e., H varies as $H(t)$ over the time span of the data. It should be realized that *mBm* is also multifractal in nature due to multiple fractal dimensions in the system within the time span of the data. Tunability of its local regularity is a valuable property of *mBm*, realizing which there has been increased interest in modeling various geophysical systems as *mBm* [21–23].

Although there is increasing use of DFA as a technique to study LRD in time series data, it is widely recognized that it yields a single Hurst exponent, and thus cannot distinguish between multi-fractal and mono-fractal systems, e.g., between mBm and fBm. In fact most systems exhibit time varying H exponent, but the estimates yield a constant value. Further, previous studies show the effect of data size used on the Hurst exponent [24,25], thus requiring caution in the interpretation of the estimated values. This is in direct agreement with our study of effect of data size on the Hurst exponent estimated by *DFA* in *mBm* data as seen in Section 4.2. Other schemes such as Multi Fractal Detrended Fluctuation Analysis (*MF-DFA*) were proposed [26], though such techniques address the multifractal nature of time series with respect to one fractal dimension at a time and do not provide a solution with respect to estimating the time varying fractal structure of *mBm*. It is apparent that *DFA* and other similar techniques were assumed to locally estimate a time averaged Hurst exponent [27,28]. This assumption underlies estimator techniques with sliding windows. We believe the success of estimating such a time average depends on the assumption of local linearity of the Hurst exponents, and is analyzed in detail in Section 3.

Using *mBm* data generated from linearly varying Hurst exponents, $H(t)$, we show that *DFA* in fact estimates the time average of $H(t)$ and test the dependence of estimated exponent on various data and technique related parameters. The primary motivation for our study is to establish a bench mark for the performance of *DFA* in estimating a time averaged Hurst exponents from *mBm* data, and, identify its limits.

Sections 2 and 3 introduce preliminary ideas of *LRD*, *fBm*, *mBm*, and, *DFA*, and, establish our estimation methodology. The main results of the study and conclusions follow in Sections 4 and 5, respectively.

2. Fractional/multifractional Brownian motion

Long Range Dependence (*LRD*), commonly identified as self-affinity, self-similarity, or long-range persistence, is a statistical property of time series data where the rate of decay of its autocovariance is slower than exponential, and most commonly a power law. This property is usually quantified using the Hurst exponent H (also known as the Hölder exponent), which is measured using *R/S* or the fluctuation analysis (*FA*) technique for stationary data. The Hurst exponent, $H \in (0, 1)$ with increasing value implying increasing *LRD*, and 0.5 as the threshold where the correlations are completely absent.

Fractional Gaussian noise (*fGn*, which is stationary in nature) is proposed as a model for data with *LRD* and fractional Brownian motion (*fBm*) (its non-stationary counterpart) is its corresponding Wiener process generated using *fGn* as its incremental process. A continuous time fractional Brownian motion (*fBm*), $B_H(t)$ with Hurst exponent H is a Gaussian process with zero-mean and is H-self affine i.e.,

$$B_H(\lambda t) \cong \lambda^H B_H(t), \quad \forall\, \lambda > 0. \tag{1}$$

Also, its covariance varies by definition as [29],

$$\mathrm{cov}[B_H(t_1), B_H(t_2)] = \frac{1}{2}(|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}). \tag{2}$$

Thus, H characterizes the relative smoothness of the resulting Brownian motions. It can also be seen that when $H = 1/2$ and $t_1 > t_2$, $\mathrm{cov}[B_H(t_1), B_H(t_2)] = t_2$, thus it is a Wiener process (Brownian motion). However when $H > 1/2$, then the increments are positively correlated and when $H < 1/2$, the increments are negatively correlated. This means that we have a smooth long-term correlated time series data for when $H > 1/2$ and anti-correlated data for when $H < 1/2$. The increments in a fBm is fractional Gaussian noise (*fGn*) as seen in Eq. (3), and is stationary in nature.

$$G_H(t) = B_H(t + 1) - B_H(t). \tag{3}$$

A problem with *fBm* is that although they capture the self-similarities well, the pointwise irregularity given by the constant Hurst parameter, H, is invariant in time. This restricting condition can be over come by generalizing *fBm* to a