Physica A 419 (2015) 385–394

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Stochastic modeling of stock price process induced from the conjugate heat equation

Seong-Hun Paeng

Department of Mathematics, Konkuk University, 1 Hwayang-dong, Gwangjin-gu, Seoul 143-701, Republic of Korea

HIGHLIGHTS

- A new stochastic model of stock prices is induced by the conjugate heat equation.
- In our model, the volatility term is affected by inflation and exchange rate.
- Our model modifies the Black–Scholes equation.

ARTICLE INFO

Article history:
Received 20 May 2014
Received in revised form 7 August 2014
Available online 26 September 2014

Keywords:
Time-dependent metric
Conjugate heat equation
Brownian motion
The Black–Scholes equation

ABSTRACT

Currency can be considered as a ruler for values of commodities. Then the price is the measured value by the ruler. We can suppose that inflation and variation of exchange rate are caused by variation of the scale of the ruler. In geometry, variation of the scale means that the metric is time-dependent. The conjugate heat equation is the modified heat equation which satisfies the heat conservation law for the time-dependent metric space. We propose a new model of stock prices by using the stochastic process whose transition probability is determined by the kernel of the conjugate heat equation. Our model of stock prices shows how the volatility term is affected by inflation and exchange rate. This model modifies the Black–Scholes equation in light of inflation and exchange rate.

1. Introduction

The Black–Scholes equation is an option pricing equation obtained under the assumption that the stock price S_t follows the stochastic process

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1.1}$$

where μ is the increasing rate of S_t, σ is the volatility constant and B_t is the Brownian motion [1,2]. In this model, inflation and exchange rate are not considered. By conversations with economists about this topic, we realized that they use real stock prices instead of nominal stock prices in order to reflect the effect of inflation. Precisely, it is assumed that the real stock price $\tilde{S}_t$ follows the stochastic process

$$d\tilde{S}_t = \tilde{\mu}\tilde{S}_t dt + \sigma\tilde{S}_t dB_t, \tag{1.2}$$

where $\tilde{\mu}$ is the increasing rate of $\tilde{S}_t$. Then the Black–Scholes equation is modified with the inflation rate as we will see in (4.20). Even if we assume the model (1.2), the volatilities of nominal stock prices are still not affected by inflation as we will

E-mail address: shpaeng@konkuk.ac.kr.

http://dx.doi.org/10.1016/j.physa.2014.09.021
0378-4371/

see in (4.19). (Also see Ref. [3].) In Section 5, we give evidences which show that the volatility is affected by inflation and exchange rate. Hence the existing model (1.2) is not consistent with reality.

We start from the insight that currency can be considered as a ruler for values of commodities. Then the price is the measured value by the ruler. In other words, price can be considered as the parametrization of values of commodities by the currency. We can suppose that inflation and variation of exchange rate are caused by variation of the scale of the ruler, which means the metric tensor is time-dependent. In Riemannian geometry, the metric tensor g is 2-tensor such that $g(v, v)$ is the square of the norm of v. More precisely, let M be an n-dimensional Riemannian manifold and (U, ϕ) be a chart around $p \in M$. We denote the coordinate functions by $(y^1, \dots, y^n)$. Then the metric tensor g can be expressed as $g = \sum_{i,j} g_{ij} \mathrm{d}y^i \otimes \mathrm{d}y^j$, shortly $g = \sum_{i,j} g_{ij} \mathrm{d}y^i \mathrm{d}y^j$. If $v = \sum_i v^i \frac{\partial}{\partial y^i}$, then $\|v\|^2 = g(v, v) = \sum_{i,j} g_{ij} v^i v^j$. In our case, if the coordinate function y is the nominal price (i.e. the parametrization of values of commodities by the currency) and m is the square of the real price of the unit currency, then the metric g can be expressed as $g = m\mathrm{d}y^2 (= m\mathrm{d}y \otimes \mathrm{d}y)$. So if $v (= v\frac{\partial}{\partial y})$ is the nominal price of a commodity A, then $\sqrt{m}v = \sqrt{g(v, v)}$ is the real price of A. (Since $\mathbb{R}$ can be considered as a vector space, we can identify a point whose coordinate is $v \in \mathbb{R}$ with the vector $v\frac{\partial}{\partial y}$.) Hence g can be considered as a function from the nominal price v to the real price $\sqrt{g(v, v)}$.

We can consider inflation as contraction of the ruler, which implies that the metric g decreases as time t increases. If the metric g is independent of time, then the heat equation is $\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u = 0$ which satisfies the heat conservation law, where Δ is the Laplacian for the metric g. If g depends on time t (i.e. $g = g(t)$), then the heat equation $\frac{\partial u}{\partial t} - \frac{1}{2}\Delta_t u = 0$ does not satisfy the heat conservation law, but the following conjugate heat equation satisfies the heat conservation law as we will see in Section 2:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_t u - hu, \tag{1.3}$$

where Δ_t is the Laplacian for the metric $g = g(t)$. In our case, let y be the coordinate function as above (i.e. nominal price at time t) and $m(t)$ be the square of the real value of the unit currency at time t. Then $g(t) = m(t)\mathrm{d}y^2$ and $h = \frac{m'}{2m}$. (See Section 2 and [4].)

In the models (1.1) and (1.2) of stock prices, the volatility terms of stock prices follow the Brownian motion. The Brownian motion is a stochastic process such that the heat kernel serves the density of the transition probability [5]. In our new model, we use a stochastic process $\mathcal{B}_t$ whose transition probability is determined by the kernel (1.5) of the conjugate heat equation (1.3) instead of the heat kernel. Hence we propose the following stochastic model of stock prices, which is our main claim:

Stochastic model of stock prices

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}\mathcal{B}_t. \tag{1.4}$$

Recall that $(\mathrm{d}B_t)^2 = \mathrm{d}t$ for the Brownian motion B_t. In our case, we prove the following theorem in Section 3:

Theorem 1. *If the metric $g(t)$ is time-dependent so that $g(t) = m(t)\mathrm{d}y^2$, then the kernel of (1.3) is*

$$p(t, x, y) = \frac{1}{\sqrt{m(t)}} \frac{1}{\sqrt{2\pi\beta(t)}} \exp\Big(-\frac{(x-y)^2}{2\beta(t)}\Big), \tag{1.5}$$

where $\beta(t) = \int_0^t m(s)^{-1}\mathrm{d}s$. Furthermore, we obtain that

$$(\mathrm{d}\mathcal{B}_t)^2 = \frac{1}{m(t)}\mathrm{d}t.$$

Our new Black–Scholes equation induced from Theorem 1 is as follows.

Theorem 2. *Let the stock price S_t follow $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}\mathcal{B}_t$ for constants μ, σ. Let $V(S, t)$ be the price of the option at time t if the price of the underlying stock at time t is S. If the real value of the unit currency at time t is $\sqrt{m(t)}$, then V satisfies that*

$$rV = \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\sigma^2 S^2}{m(t)}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S}. \tag{1.6}$$

When some economists saw our model (1.4), they wondered if our model (1.4) is the same as the model (1.2). Comparing our Black–Scholes equation (1.6) with the Black–Scholes equation (4.20) induced from (1.2), we can verify that our model (1.4) is completely different from (1.2). Also we show in Section 5 that our model seems to be more consistent with reality.

For (1.6), we should determine $m(t)$ which is the square of the real value of the unit currency at time t. If we let $P(t)$ be the price index at time t, then the inflation rate $I(t)$ is $I(t) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}P(t)}{P(t)}$ and $P(t) = P(t_0)e^{\int_{t_0}^t I\mathrm{d}s}$ for the reference time $t_0 \le 0$. Since the reference time is t_0, we have $m(t_0) = 1$. In domestic trade, we have

$$m(t) = \frac{P(t_0)^2}{P(t)^2}.$$