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Comparing numerical integration schemes for time-continuous car-following models

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HIGHLIGHTS

- We propose novel performance metrics for numerical integration schemes.
- For car-following models, the ballistic scheme is always superior to Euler's scheme.
- The standard RK4 scheme is only efficient for unperturbed single-lane traffic.
- Heun's scheme is generally the best for simple situations.
- The ballistic scheme prevails for complex situations with stops and lane changes.

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ABSTRACT

When simulating trajectories by integrating time-continuous car-following models, standard integration schemes such as the fourth-order Runge–Kutta method (RK4) are rarely used while the simple Euler method is popular among researchers. We compare four explicit methods both analytically and numerically: Euler's method, ballistic update, Heun's method (trapezoidal rule), and the standard RK4. As performance metrics, we plot the global discretization error as a function of the numerical complexity. We tested the methods on several time-continuous car-following models in several multi-vehicle simulation scenarios with and without discontinuities such as stops or a discontinuous behavior of an external leader. We find that the theoretical advantage of RK4 (consistency order 4) only plays a role if both the acceleration function of the model and the trajectory of the leader are sufficiently often differentiable. Otherwise, we obtain lower (and often fractional) consistency orders. Although, to our knowledge, Heun's method has never been used for integrating car-following models, it turns out to be the best scheme for many practical situations. The ballistic update always prevails over Euler's method although both are of first order.

1. Introduction

Time-continuous car-following models (or more precisely, their longitudinal dynamics components) prescribe the acceleration of individual cars as a function of the driver's characteristic behavior and the surrounding traffic. Formally, their mathematical formulation is equivalent to that of physical particles following Newtonian dynamics with the physical forces

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replaced by "social forces" [1]. In contrast to car-following models formulated in discrete time (coupled maps) or fully discretely (cellular automata), time-continuous car-following models must be augmented with a numerical integration method in all but the most trivial analytically solvable cases [2].

Mathematically speaking, time-continuous car-following models without explicit reaction time delay represent coupled ordinary differential equations (ODE). Some often investigated models of this class include the optimal-velocity model (OVM) of Ref. [3], derivatives such as the full-velocity difference model (FVDM) of Ref. [4], and the Intelligent-Driver Model (IDM) by Treiber et al. [5]. Early car-following models such as that by Gazis, Herman and Rothery (GHR, [6]) or the linear adaptive cruise control(ACC) model by Helly [7] also fall into this class.

Because of the complexity and possible event-oriented components of traffic flow scenarios, one generally assumes a fixed common time step h and explicit numerical schemes to obtain trajectories from the model equations. In the general literature on numerical mathematics (see, e.g., Ref. [8]), the standard explicit numerical integration scheme for ODEs is the fourth-order Runge–Kutta method (RK4). However, in the domain of microscopic traffic flow modeling, the use of this method is rarely stated (counterexamples include [9,10]). Instead, most authors apply simpler methods or do not specify the numerical method at all. Commonly used schemes are the simple Euler method [11] or the ballistic update assuming constant accelerations during one time step [2]. Notice that also the open-source traffic simulators SUMO [12] and AIMSUN [13] use simple Euler update for the positions.

Many car-following models are formulated in discrete time by mapping the positions and speeds of the vehicles at time t to these variables at time $t + h$. Generally, such models are mathematically identical to a time-continuous car-following model combined with a certain update procedure. For example, in the congested regime, Newell's microscopic model [14] maps the position of vehicle i at time $t + h$ to the position of the leading vehicle $i - 1$ at time t minus the (effective) vehicle length. As shown in Ref. [2], this is mathematically equivalent to integrating the OVM by a simple Euler step of size h provided the OVM's optimal velocity function corresponds to a triangular fundamental diagram with a maximum density equal to the inverted vehicle length and car-following time gaps equal to h. Similarly, a time-discrete model has been derived from the GHR model by using Euler update with h equal to the reaction time, and qualitatively different behavior has been found compared to integrating the original GHR model with the RK4 method [15]. In the context of car-following methods, the ballistic method is particularly appealing since it allows to model reaction times without introducing explicit delays which would transform the ODEs of time-continuous car-following models (such as all time-continuous models mentioned above) into delay-differential equations. It has been shown [16] that integration of the IDM by the ballistic method with time step h is essentially equivalent to an explicit reaction time delay $T_r = h/2$ of the corresponding delay-differential equations (which are, then, integrated by higher-order methods or very small time steps).

Nevertheless, it is often desired to approach the true solution of time-continuous car-following models as closely as possible. A criterion for the quality of an integration scheme is its (local or global) consistency order stating how fast the approximate numerical solution converges to the true solution when decreasing the time step h ([8], see Section 2 for details). However, for practical integration steps h, higher-order methods do not necessarily lead to lower discretization errors. Moreover, if the acceleration function of the model is not sufficiently smooth (differentiable) or the simulation scenario contains discontinuities such as stops, lane changes, or traffic lights, the actual consistency order of a given numerical scheme is generally lower than its nominal order [8]. Finally, higher-order methods need several evaluations of the model's acceleration function per vehicle and per time step while Euler's method and the ballistic scheme need only one.

This leads to the following question: "Does the higher numerical accuracy of higher-order schemes outweigh their higher numerical complexity in terms of computation time, for practical cases?" Specifically, we would like to know which numerical scheme has the lowest global discretization error for a given numerical complexity, and how this depends on the model and the simulation scenario.

In this work, we profile four numerical methods, simple Euler, ballistic scheme, Heun's rule or trapezoidal rule, and RK4, for three car-following models (OVM, FVDM, IDM) in several multi-vehicle simulation scenarios. We found that RK4 is, in fact, superior if certain rather restrictive conditions for the differentiability of the acceleration function and the external data (in our case, the leader's trajectory) are satisfied, and if a high numerical precision is required. In most practical situations, however, the ballistic scheme and the trapezoidal rule turn out to be the most efficient and robust methods, although the latter is rarely used. Moreover, the ballistic update always prevails over simple Euler although both are of first order.

In the next two sections, we specify the integration schemes in the context of car-following models and give an analytical analysis of the truncation errors. In Section 4, we describe the simulation tests, define the numerical complexity as a measure for the computational burden, and the discretization error in terms of a vector norm on the deviations of the trajectories. In Section 5, we present the simulations and results. Finally. Section 6 gives a discussion and an outlook.

2. Integration schemes for car-following models and their mathematical properties

2.1. Mathematical formulation

We start by writing the dynamics created by time-continuous car-following models without explicit reaction time as a general system of ordinary differential equations,

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = \vec{f}(\vec{y}, t). \tag{1}$$