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Numerical determination of hitting time distributions from their Laplace transforms: One dimensional diffusions

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HIGHLIGHTS

- The maximum entropy method is used to invert a Laplace transform.
- Only eight real values of the transformed parameter are needed.
- The Laplace transform is determined up to errors.
- We propose extensions of the maximum entropy methods to cope with errors in the data.
- The extension are useful for standard applications of the maximum entropy method.

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ABSTRACT

In a previous paper we studied a method to determine the probability density of barrier crossing times by a Brownian motion from the knowledge of its Laplace transform. This knowledge combined with the method of maximum entropy yields quite good reconstructions. The aim of this work is to extend the previous analysis in two directions. On one hand, we consider diffusions with non constant coefficients. This forces us to determine the Laplace transform numerically or by means of simulations. On the other hand, and this is the gist of this note, as numerical problems involve errors, we consider as well two possible extensions of the maximum entropy procedure which allow us to incorporate those errors into the probability reconstruction process.

1. Introduction and preliminaries

In our previous note, [1], we proposed a method for determining the probability density of a first exit time, based on the possibility of computing its Laplace transform given by

$$\phi(\alpha) \equiv E[e^{-\alpha T}] = \int_0^\infty e^{-\alpha t} dF_T(t)$$

or to introduce some notation, to determine $F_T(t)$ or its density $f_T(t)$ from the knowledge of

$$\phi(\alpha, x) \equiv E^x[e^{-\alpha T}] = \int_0^\infty e^{-\alpha t} dF_T(t), \quad \text{for } x \in S. \tag{1}$$

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The $\phi(\alpha, x)$ are obtained by solving a one dimensional boundary problem. In (1) $\alpha > 0$ is a real number and $T : \Omega \to [0, \infty]$ is, to be as general as possible, any positive random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. The variable x refers to the starting point of the trajectories of some underlying Markov process X which we shall describe below. At this point we only add that the maxentropic techniques that we shall describe below can be used to determine the distribution of any positive random variable from the value of its Laplace transform, regardless of it being a hitting time of some process or not.

We direct the reader to Ref. [1] for a list of references with applications to a variety of fields, as well as for references to previous work on the two lines of work related to the underlying subject matter of this paper, namely the inversion of Laplace transforms.

To bring in the use of maximum entropy techniques, we recast the Laplace inversion problem as a fractional moment problem by introducing $Y = e^{-T}$, which renders the following representation for $\phi(\alpha, x)$.

$$\phi(\alpha, x) \equiv E^x[Y^\alpha] = \int_0^1 y^\alpha dF_Y^x(y), \quad \text{for } x \in S, \tag{2}$$

which is the moment curve of the [0, 1] valued random variable Y. Once we have found a possible probability density $f_Y(y, x)$ of Y, to obtain the corresponding density for T we perform the change of variables $f_T(t, x) = e^{-t} f_Y(e^{-t}, x)$. Having said this, we want to solve the problem

$$\text{Determine a density } f_Y(y, x) \quad \text{on } [0, 1] \text{ given } \phi(a_i, x)\ i = 1, \ldots, M. \tag{3}$$

The thrust of this paper is to introduce two extensions of the maximum entropy method used in Ref. [1], to deal with the problem when the knowledge of $\phi(\alpha, x)$ is approximate.

The paper is organized as follows. In the remaining part of this section we detail the statement of the problem. First we describe the computation of the Laplace transform of the probability density of a hitting time in the two different examples that we are going to study numerically. To begin with, we consider the hitting time of a sphere starting from a point in its exterior, and then the hitting time of the boundary of an interval by an Ornstein–Uhlenbeck (OU) process, and for that we rapidly recall some of its properties.

Section 2 is devoted to the numerical computation of $\phi(\alpha, x)$ for the hitting time of the boundary of an interval by the OU process in two different ways. First by numerically solving a boundary problem, and then by Monte Carlo simulation. These results will be the inputs for the maximum entropy methods. In Section 3 we present two possible extensions of the maxentropic procedure to deal with the case of observational or estimation errors. As mentioned, this section can be regarded as the important part of the paper. Which extension to consider shall depend on the information about the measurement or estimation errors.

Then, Section 4 is devoted to a numerical implementation of the two maxentropic procedures having the results of the computational results described in Section 2 as input. A final section summing up the results comes after that.

1.1. Hitting a sphere from the exterior

This is an interesting example, related to probabilistic methods in wave propagation. See Refs. [2] or [3] for an extension of those results. Using a martingale argument it was proved there (with a slightly different notation) that if $B(t)$ denotes a d-dimensional Brownian motion, S the sphere of radius R centered at the origin, and $x \in \mathbb{R}^d$ with $|x| > R$, and $T = \inf\{t > 0 : |B(t)| < R\}$, then

$$E^x[e^{-\alpha T}] = \left(\frac{R}{|x|}\right)^h \frac{K_h(\sqrt{2\alpha}|x|)}{K_h(\sqrt{2\alpha}R)} \tag{4}$$

where $h = (d-2)/2$, and K_h stands for the modified Bessel function of the second type of order h. See the short appendix to this section for an explanation When $d = 3$, due to the simple form of the $K_{1/2}$ we have

$$E^x[e^{-\alpha T}] = \frac{R}{|x|} e^{-\sqrt{2\alpha}(|x|-R)} \tag{5}$$

which is as exact and simple to invert as one can get, but when $d = 2$ we have

$$E^x[e^{-\alpha T}] = \frac{K_0(\sqrt{2\alpha}|x|)}{K_0(\sqrt{2\alpha}R)} \tag{6}$$

where K_0 is tabulated to high precision in many computational platforms, but as we want to illustrate the use of the maximum entropy method when there is error in the data, we shall consider just a couple of terms approximation to K_0 based on the Hankel expansion, namely

$$K_0(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 - \frac{1}{8z} + \frac{9}{2(8z)^2}\right). \tag{7}$$

We should add, that even though technically speaking, these two problems are not 1-dimensional, actually, due to the intrinsic rotational symmetries in both cases, they are one dimensional diffusion processes. In both cases the radial part of the three and two dimensional underlying Brownian motions are infinitesimal generators described by a Bessel type equation.